Solutions HW 13

9.4.2 Write the given system in matrix form $x' = Ax + f$

$$r'(t) = 2r(t) + \sin t$$
$$\theta'(t) = r(t) - \theta(t) + 1$$

We write this as

$$\begin{pmatrix} r'(t) \\ \theta'(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} r(t) \\ \theta(t) \end{pmatrix} + \begin{pmatrix} \sin(t) \\ 1 \end{pmatrix}$$

9.4.4 Write the given system in matrix form $x' = Ax + f$

$$\frac{dx}{dt} = x + y + z$$
$$\frac{dy}{dt} = 2x - y + 3z$$
$$\frac{dz}{dt} = x + 5z$$

We write this as

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 1 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

9.4.8 Rewrite $\frac{d^3y}{dt^3} - \frac{dy}{dt} + y = \cos(t)$ as a first order system in normal form.

Note that the equation says that $\frac{d^3y}{dt^3} = \frac{dy}{dt} - y + \cos(t)$. Setting $x_1 = y$, $x_2 = \frac{dy}{dt}$, $x_3 = \frac{d^2y}{dt^2}$, (so $\frac{d^3y}{dt^3} = x_2 - x_1 + \cos(t)$) we get

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \frac{dy}{dt} \\ \frac{d^2y}{dt^2} \\ \frac{d^3y}{dt^3} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_2 - x_1 + \cos(t) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cos(t) \end{pmatrix}$$

9.4.10 Write the given system as a set of scalar equations

$$x' = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} x \quad + \quad e^t \begin{pmatrix} t \\ 1 \end{pmatrix}$$

This becomes the equations

$$\begin{aligned} x_1' &= 2x_1 + x_2 + te^t \\ x_2' &= -x_1 + 3x_2 + e^t \end{aligned}$$

9.4.16 Determine whether the given vector functions are linearly dependent or independent on the interval $(-\infty, \infty)$

$$\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$$

We compute the Wronskian

$$\det \begin{pmatrix} \sin t & \sin 2t \\ \cos t & \cos 2t \end{pmatrix} = \sin t \cos 2t - \sin 2t \cos t = -\sin t$$

where the last step can be deduced by using trig identities. Since $-\sin t$ is not identically 0, the vector functions are linearly independent. (Alternatively, one can check that the Wronksian is nonzero at a point such as $t = \frac{\pi}{2}$.)

9.4.18 Determine whether the given vector functions are linearly dependent or independent on the interval $(-\infty, \infty)$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} t \\ 0 \\ t \end{pmatrix}, \begin{pmatrix} t^2 \\ 0 \\ t^2 \end{pmatrix}$$

These functions are linearly independent, since a linear relations requires finding nonzero **constants** c_1, c_2, c_3 such that $c_1 + c_2 t + c_3 t^2 = 0$. But $1, t, t^2$ are linearly independent, so no such constants exist.

Note that even though the vector functions are linearly independent, their Wronksian is still zero.

9.4.22 Determine whether the given functions form a fundamental solution set to an equation $x'(t) = Ax$. If they do, find a fundamental matrix for the system and give a general solution.

$$x_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, \quad x_2 = \begin{pmatrix} \sin t \\ \cos t \\ -\sin t \end{pmatrix}, \quad x_3 = \begin{pmatrix} -\cos t \\ \sin t \\ \cos t \end{pmatrix}$$

We start by computing the Wronksian

$$\det \begin{pmatrix} e^t & \sin t & -\cos t \\ e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \end{pmatrix} = e^t(\cos^2 t+\sin^2 t)-e^t(\sin t\cos t-\sin t\cos t)+e^t(\sin^2 t+\cos^2 t) = 2e^t$$

Since this is nowhere 0, the solutions are linearly independent and form a fundamental set. A fundamental matrix is

$$\begin{pmatrix} e^t & \sin t & -\cos t \\ e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \end{pmatrix}$$

and a general solution is $c_1x_1 + c_2x_2 + c_3x_3$.

9.4.24 Verify that the vector functions

$$x_1 = \begin{pmatrix} e^{3t} \\ 0 \\ e^{3t} \end{pmatrix}, \quad x_2 = \begin{pmatrix} -e^{3t} \\ e^{3t} \\ 0 \end{pmatrix}, \quad x_3 = \begin{pmatrix} -e^{-3t} \\ -e^{-3t} \\ e^{-3t} \end{pmatrix}$$

are solutions to the homogenous system

$$x' = Ax = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} x,$$

on $(-\infty, \infty)$ and that

$$x_p = \begin{pmatrix} 5t+1 \\ 2t \\ 4t+2 \end{pmatrix}$$

is a particular solution to

$$x' = Ax + \begin{pmatrix} -9t \\ 0 \\ -18t \end{pmatrix} = Ax + f(t)$$

Find a general solution to $x' = Ax + f(t)$.

We check directly that

$$x_1' = \begin{pmatrix} 3e^{3t} \\ 0 \\ 3e^{3t} \end{pmatrix} = Ax_1$$
$$x_2' = \begin{pmatrix} -3e^{3t} \\ 3e^{3t} \\ 0 \end{pmatrix} = Ax_2$$
$$x_3' = \begin{pmatrix} 3e^{-3t} \\ 3e^{-3t} \\ -3e^{-3t} \end{pmatrix} = Ax_3$$
$$x_p' = \begin{pmatrix} 5 \\ 2 \\ 4 \end{pmatrix} = Ax_p + f(t)$$

A general solution to $x' = Ax + f(t)$ is $c_1x_1 + c_2x_2 + c_3x_3 + x_p$.

9.4.25 Prove that the operator $L[x] = x' - Ax$ is a linear operator.

We must show $L[x+y] = L[x] + L[y]$ and $L[cx] = cL[x]$.

$L[x+y] = (x+y)' - A(x+y) = x' + y' - Ax - Ay = (x' - Ax) + (y' - Ay) = L[x] + L[y]$

$L[cx] = (cx)' - A(cx) = cx' - cAx = c(x' - Ax) = cL[x]$

9.4.26 Let $X(t)$ be a fundamental matrix for the system $x' = Ax$. Show that $x(t) = X(t)X^{-1}(t_0)x_0$ is the solution to the initial value problem $x' = Ax, x(t_o) = x_0$.

Since $x(t)$ is a linear combination of the columns of the fundamental matrix, we just need to check that it satisfies the initial conditions. But $x(t_0) = X(t_0)X^{-1}(t_0)x_0 = Ix_0 = x_0$ as desired, so $x(t)$ is the dersired solutions.

9.5.6 Find eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

We start by computing the characteristic polynomial.

$$\det \begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = -\lambda^3 + 3\lambda + 2 = (2-\lambda)(1+\lambda)^2$$

So the eigenvalues are 2 and -1.

For $\lambda = -1$ we must find the kernel of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Row reducing we get

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which gives eigenvectors

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda = 2$ we must find the kernel of

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Row reducing we get

$$\begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

which gives the eigenvector

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

9.5.10 Find all eigenvalues and eigenvectors of

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

We start by computing the characteristic polynomial.

$$\det \begin{pmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{pmatrix} = (1-\lambda)(\lambda^2 - 2\lambda + 2)$$

The first factor gives eigenvalue 1, the second gives eigenvalues $1 \pm i$.

For $\lambda = 1$, we must find the kernel of

$$\det \begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

which gives the eigenvector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

For $\lambda = 1 - i$ we must find the kernel of

$$\det \begin{pmatrix} i & 2 & -1 \\ 0 & i & 1 \\ 0 & -1 & i \end{pmatrix}$$

Solving this we get the eigenvector

$$\begin{pmatrix} -2+i \\ i \\ 1 \end{pmatrix}$$

Taking conjugates, we get that the eigenvector for $\lambda = 1 + i$ is

$$\begin{pmatrix} -2-i \\ -i \\ 1 \end{pmatrix}$$

9.5.14 Find a general solution to the equation $x' = Ax$ where

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{pmatrix}$$

We start by computing the characteristic polynomial.

$$\det \begin{pmatrix} -1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 3 & -1-\lambda \end{pmatrix} = -(\lambda^3 - 7\lambda - 6) = -(\lambda+1)(\lambda+2)(\lambda-3)$$

So the eigenvalues are $-1, -2, 3$.

For $\lambda = -1$, we must find the kernel of

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 3 & 0 \end{pmatrix}$$

Row reducing we get

$$\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which gives the eigenvector

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda = -2$, we must find the kernel of

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 3 & 1 \end{pmatrix}$$

Row reducing we get

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

which gives the eigenvector

$$\begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

For $\lambda = 3$, we must find the kernel of

$$\begin{pmatrix} -4 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -4 \end{pmatrix}$$

Row reducing we get

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

which gives the eigenvector

$$\begin{pmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{pmatrix}$$

Combining these, we get that the general solution to the differential equation is

$$c_1 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} \frac{1}{3} \\ \frac{4}{3} \\ 1 \end{pmatrix}$$

9.5.20 Find a fundamental matrix for the system x'=Ax, where

$$A = \begin{pmatrix} 5 & 4 \\ -1 & 0 \end{pmatrix}$$

The characteristic polynomial of A is $\lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$, so the eigenvalues are $\lambda = 1, 4$. For $\lambda = 1$ we must find the kernel of

$$\begin{pmatrix} 4 & 4 \\ -1 & -1 \end{pmatrix}$$

which is spanned by

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

For $\lambda = 4$ we must find the kernel of

$$\begin{pmatrix} 1 & 4 \\ -1 & -4 \end{pmatrix}$$

which is spanned by

$$\begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

The corresponding fundamental matrix is

$$\begin{pmatrix} -e^t & -4e^{4t} \\ e^t & e^{4t} \end{pmatrix}$$

9.5.26 Find a general solution to the system of equations

$$x' = 3x - 4y$$
$$y' = 4x - 7y$$

This system can be rewritten as $x' = Ax$, where

$$A = \begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix}$$

The characteristic polynomial is $\lambda^2 + 4\lambda - 5 = (\lambda - 1)(\lambda + 5)$, so the eigenvalues are 1 and -5. For $\lambda = 1$ we must find the kernel of

$$\begin{pmatrix} 2 & -4 \\ 4 & -8 \end{pmatrix}$$

which is spanned by

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda = -5$ we must find the kernel of

$$\begin{pmatrix} 8 & -4 \\ 4 & -2 \end{pmatrix}$$

which is spanned by

$$\begin{pmatrix}1\\2\end{pmatrix}$$

Combining these, we see the general solution to the initial system is $x = 2c_1e^t + c_2e^{-5t}$, $y = c_1e^t + 2c_2e^{-5t}$.

9.5.34 Solve the initial value problem

$$x'(t) = \begin{pmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{pmatrix} x, \quad x(0) = \begin{pmatrix}-1\\4\\0\end{pmatrix}$$

From the eigenvectors and eigenvalues from problem 6, the general solution to this equation is

$$x(t) = c_1e^{-t}\begin{pmatrix}-1\\1\\0\end{pmatrix} + c_2e^{-t}\begin{pmatrix}-1\\0\\1\end{pmatrix} + c_3e^{2t}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Plugging in the initial condition, we must solve the equations

$$\begin{pmatrix}-1 & -1 & 1\\1 & 0 & 1\\0 & 1 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}-1\\4\\0\end{pmatrix}$$

Row reducing the system and backsolving gives, $c_1 = 3, c_2 = -1, c_3 = 1$, so the desired solution is

$$x(t) = 3e^{-t}\begin{pmatrix}-1\\1\\0\end{pmatrix} - e^{-t}\begin{pmatrix}-1\\0\\1\end{pmatrix} + c_3e^{2t}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

9.5.35 a. Show that the matrix

$$A = \begin{pmatrix}1 & -1\\4 & -3\end{pmatrix}$$

has a repeated eigenvalue, and only one eigenvector.

The characteristic polynomial is $\lambda^2+2\lambda+1 = (\lambda+1)^2$, so the only eigenvalue is $\lambda = -1$. Searching for eigenvectors, we must find the kernel of

$$\begin{pmatrix}2 & -1\\4 & -2\end{pmatrix}$$

which is spanned by

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

b. Use your answer to part a. to find a nontrivial solution to $x' = Ax$.

$$e^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

c. Try to find a second solution of the form $te^{-t}u_1 + e^{-t}u_2$.

Plugging this expression into $x' = Ax$, we get

$-te^{-t}u_1 + e^{-t}u_1 - e^{-t}u_2 = te^{-t}Au_1 + e^{-t}Au_2$.

Grouping the e^{-t} and te^{-t} terms together, we get to vector relations

$-u_2 + u_1 = Au_2$ or $(A+I)u_2 = u_1$

and $-u_1 = Au_1$, or $(A+I)u_1 = 0$.

We want u_1 to be an eigenvector. To find u_2, we can either solve the given set of linear equations, or just guess a u_2 and see if $(A+I)u_2$ is an eigenvector. (This may seem ad hoc, but it works as long as your guess for u_2 is not already an eigenvector.) If we guess

$$u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

then

$$(A+I)u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

which is an eigenvector. So we get a solution of the differential equation

$$e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-t}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

d. What is $(A+I)^2u_2$?

$(A+I)^2u_2 = (A+I)(A+I)u_2 = (A+I)u_1 = 0$ from the equations we derived in part c.

9.5.36 Use the method of problem 35 to find a general solution to the system

$$x'(t) = \begin{pmatrix} 5 & -3 \\ 3 & -1 \end{pmatrix} x(t)$$

Computing the characteristic polynomial, we get that $\lambda = 2$ is a double root, and

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector, so $e^{2t}v_1$ is a solution to the differential equation.

As in problem 35, we guess a solution of the form $te^{2t}u_1 + e^{2t}u_2$. This gives rise to the equations $(A - 2I)u_1 = 0$, $(A - 2I)^2 u_2 = u_1$.

Guessing

$$u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we get

$$u_1 = \begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

which is an eigenvector. So another solution is $te^{2t}u_1 + e^{2t}u_2$. Combining these, we get a general solution

$$c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 (te^{2t} \begin{pmatrix} 3 \\ 3 \end{pmatrix} + e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix})$$