UCB Math 228A, Fall 2014: Homework Set 3

Due Oct. 20, 2014

1. The m-file iter_bvp_Asplit.m implements the Jacobi, Gauss-Seidel, and SOR matrix splitting methods on the linear system arising from the boundary value problem $u''(x) = f(x)$ in one space dimension.

   (a) Run this program for each method and produce a plot similar to Figure 4.2.

   (b) The convergence behavior of SOR is very sensitive to the choice of ω (`omega` in the code). Try changing from the optimal ω to $\omega = 1.8$ or 1.95.

   (c) Let $g(\omega) = \rho(G(\omega))$ be the spectral radius of the iteration matrix G for a given value of ω. Write a program to produce a plot of $g(\omega)$ for $0 \leq \omega \leq 2$.

   (d) From equations (4.22) one might be tempted to try to implement SOR as

   ```
   for iter=1:maxiter
      uGS = (DA - LA) \ (UA*u + rhs);
      u = u + omega * (uGS - u);
      end
   ```

   where the matrices have been defined as in iter_bvp_Asplit.m. Try this computationally and observe that it does not work well. Explain what is wrong with this and derive the correct expression (4.24).

2. (a) The Gauss-Seidel method for the discretization of $u''(x) = f(x)$ takes the form (4.5) if we assume we are marching forwards across the grid, for $i = 1,\ 2,\ \ldots,\ m$. We can also define a *backwards Gauss-Seidel method* by setting

   $$u_i^{[k+1]} = \frac{1}{2}(u_{i-1}^{[k]} + u_{i+1}^{[k+1]} - h^2 f_i), \quad \text{for } i = m,\ m-1,\ m-2,\ \ldots,\ 1. \tag{1}$$

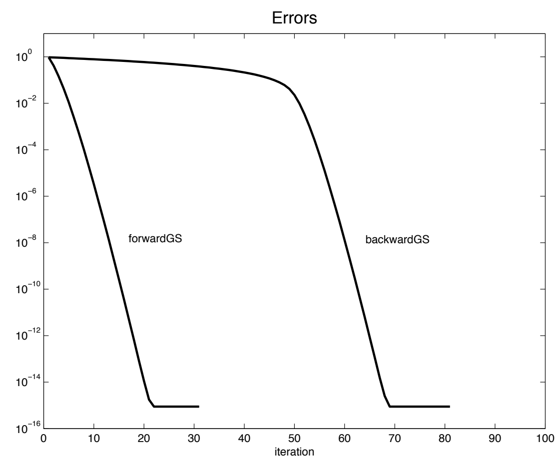
   Show that this is a matrix splitting method of the type described in Section 4.2 with $M = D - U$ and $N = L$.

   (b) Implement this method in iter_bvp_Asplit.m and observe that it converges at the same rate as forward Gauss-Siedel for this problem.

   (c) Modify the code so that it solves the boundary value problem

   $$\epsilon u''(x) = au'(x) + f(x), \qquad 0 \leq x \leq 1, \tag{2}$$

with $u(0) = 0$ and $u(1) = 0$, where $a \geq 0$ and the $u'(x_i)$ term is discretized by the one-sided approximation $(U_i - U_{i-1})/h$. Test both forward and backward Gauss-Seidel for the resulting linear system. With $a = 1$ and $\epsilon = 0.0005$. You should find that they behave very differently:



Explain intuitively why sweeping in one direction works so much better than in the other.

Hint: Note that this equation is the steady equation for an advection-diffusion PDE $u_t(x,t) + au_x(x,t) = \epsilon u_{xx}(x,t) - f(x)$. You might consider how the methods behave in the case $\epsilon = 0$.

3. Prove that the ODE

$$u'(t) = \frac{1}{t^2 + u(t)^2}, \quad \text{for } t \geq 1$$

has a unique solution for all time from any initial value $u(1) = \eta$.

4. Let $f(u) = \log(u)$.

(a) Determine the best possible Lipschitz constant for this function over $2 \leq u < \infty$.

(b) Is $f(u)$ Lipschitz continuous over $0 < u < \infty$?

(c) Consider the initial value problem

$$u'(t) = \log(u(t)),$$
$$u(0) = 2.$$

Explain why we know that this problem has a unique solution for all $t \geq 0$ based on the existence and uniqueness theory described in Section 5.2.1. (Hint: Argue that f is Lipschitz continuous in a domain that the solution never leaves, though the domain is not symmetric about $\eta = 2$ as assumed in the theorem quoted in the book.)