UCB Math 228A, Fall 2014: Homework Set 2

Due September 29, 2014

1. The matlab script `poisson.m` solves the Poisson problem on a square $m \times m$ grid with $\Delta x = \Delta y = h$, using the 5-point Laplacian. It is set up to solve a test problem for which the exact solution is $u(x,y) = \exp(x + y/2)$, using Dirichlet boundary conditions and the right hand side $f(x,y) = 1.25\exp(x + y/2)$.

   (a) Test this script by performing a grid refinement study to verify that it is second order accurate.

   (b) Modify the script so that it works on a rectangular domain $[a_x, b_x] \times [a_y, b_y]$, but still with $\Delta x = \Delta y = h$. Test your modified script on a non-square domain.

   (c) Further modify the code to allow $\Delta x \neq \Delta y$ and test the modified script.

2. (a) Show that the 9-point Laplacian (3.17) has the truncation error derived in Section 3.5. **Hint:** To simplify the computation, note that the 9-point Laplacian can be written as the 5-point Laplacian (with known truncation error) plus a finite difference approximation that models $\frac{1}{6}h^2 u_{xxyy} + O(h^4)$.

   (b) Modify the matlab script `poisson.m` to use the 9-point Laplacian (3.17) instead of the 5-point Laplacian, and to solve the linear system (3.18) where f_{ij} is given by (3.19). Perform a grid refinement study to verify that fourth order accuracy is achieved.

3. (a) Develop a spectral method for solving the two point boundary value problem defined by equations (2.64) and (2.65), for the the case where

$$a(x) = a, \quad b(x) = b, \quad \text{and} \quad c(x) = c$$

   are constants, based on the spectral method of L. Greengard, *Spectral integration and two-point boundary value problems*, SIAM J. Numer. Anal., vol. 28, pp. 1071–1080, 1991.

   (b) Solve

$$\epsilon u''(x) - u'(x) = f(x),$$

with boundary conditions

$$u(0) = \alpha, \quad \text{and} \quad u(1) = \beta,$$

where

$$\epsilon = 10^{-5}, \quad f(x) = -1, \quad \alpha = 1, \quad \beta = 3.$$

Choose $N = 15, 31, 63$, and compare your results with the exact solution (2.90).

Code Submission: E-mail all requested and supporting MATLAB files to Luming at `lwang@berkeley.edu` as a zip-file named lastname_firstname_2.zip, for example luming_wang_2.zip.