UCB Math 228A, Fall 2014: Homework Set 1

Due September 15, 2014

1. Read the paper, *Eror Analysis of the Björck-Pereyra Algorithms for Solving Vandermonde Systems*, by N. J. Higham, in *Numerische Mathematik* vol. 50, pp. 613–632, 1987. Implement a `Matlab` program for **Algorithm 2**, the dual Vandermonde solver, in this paper.

   Approximate $u^{'''}(\bar{x})$, using both **Algorithm 2** and Fornberg's method, fdcoeffF.m, downloadable from the textbook website, for $\bar{x} = 0$ and $x_j = \cos\left(\frac{(j+1/2)\pi}{n+1}\right)$ for function $u(x) = e^x$ and $n = 5, 10, 15, 20, 25$. Use `Matlab` plots to compare the accuracies achieved by these methods.

2. (a) Determine the Green's functions for the two-point boundary value problem $u''(x) = f(x)$ on $0 < x < 1$ with a Neumann boundary condition at $x = 0$ and a Dirichlet condition at $x = 1$, i.e, find the function $G(x, \bar{x})$ solving

   $$u''(x) = \delta(x - \bar{x}), \quad u'(0) = 0, \quad u(1) = 0$$

   and the functions $G_0(x)$ solving

   $$u''(x) = 0, \quad u'(0) = 1, \quad u(1) = 0$$

   and $G_1(x)$ solving

   $$u''(x) = 0, \quad u'(0) = 0, \quad u(1) = 1.$$

   (b) Using this as guidance, find the general formulas for the elements of the inverse of the matrix in equation (2.54). Write out the 5×5 matrices A and A^{-1} for the case $h = 0.25$.

3. In Example 1.4 a 3-point approximation to $u''(x_i)$ is determined based on $u(x_{i-1})$, $u(x_i)$, and $u(x_{i+1})$ (by translating from x_1, x_2, x_3 to general x_{i-1}, x_i, and x_{i+1}). It is also determined that the truncation error of this approximation is $\frac{1}{3}(h_{i-1} - h_i)u'''(x_i) + O(h^2)$, where $h_{i-1} = x_i - x_{i-1}$ and $h_i = x_{i+1} - x_i$, so the approximation is only first order accurate in h if h_{i-1} and h_i are $O(h)$ but $h_{i-1} \neq h_i$.

The program `bvp2.m` from text website is based on using this approximation at each grid point, as described in Example 2.3. Hence on a nonuniform grid the local truncation error is $O(h)$ at each point, where h is some measure of the grid spacing (e.g., the average spacing on the grid). If we assume the method is stable, then we expect the global error to be $O(h)$ as well as we refine the grid.

(a) However, if you run `bvp2.m` you should observe second-order accuracy, at least provided you take a smoothly varying grid (e.g., set `gridchoice = 'rtlayer'` in `bvp2.m`). Verify this.

(b) Suppose that the grid is defined by $x_i = X(z_i)$ where $z_i = ih$ for $i = 0,\ 1,\ \ldots,\ m+1$ with $h = 1/(m+1)$ is a uniform grid and $X(z)$ is some smooth mapping of the interval $[0, 1]$ to the interval $[a, b]$. Show that if $X(z)$ is smooth enough, then the local truncation error is in fact $O(h^2)$. Hint: $x_i - x_{i-1} \approx hX'(z_i)$.

(c) What average order of accuracy is observed on a random grid? To test this, set `gridchoice = 'random'` in `bvp2.m` and increase the number of tests done, e.g., by setting `mvals = round(logspace(1,3,50));` to do 50 tests for values of m between 10 and 1000.

4. Write a program to solve the boundary value problem for the nonlinear pendulum as discussed in the text. Find a numerical solution to this BVP with the same general behavior as seen in Figure 2.5 for the case of a longer time interval, say $T = 20$, again with $\alpha = \beta = 0.7$. Try larger values of T. What does $\max_i \theta_i$ approach as T is increased? Note that for large T this solution exhibits "boundary layers".

Code Submission: E-mail all requested and supporting MATLAB files to Luming at `lwang@berkeley.edu` as a zip-file named lastname_firstname_1.zip, for example luming_wang_1.zip.