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Math16A Sample Midterm II Solutions, Fall 2009

This is a closed book, closed notes exam. You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Look over the whole exam to find problems that you can do quickly. You need not simplify your answers unless you are specifically asked to do so. Hand in this exam before you leave.

Problem	Maximum Score	Your Score
1	20	
2	20	
3	20	
4	20	
5	10	
6	10	
Total	100	

Your Name & SID: ______________________________

Your Section & GSI: ______________________________

1. Determine the derivatives of the following functions

   (a) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$.
   Solution: $f'(x) = \frac{4x}{(x^2+1)^2}$.

   (b) $f(x) = \left(x^2 + 1\right)\sqrt{x^2 - 1}$.
   Solution: $f'(x) = \frac{x(3x^2-1)}{\sqrt{x^2-1}}$.

2. Determine the derivatives $\frac{dy}{dx}$ of the following functions at the given points

   (a) $y(x)$ as a function of x defined by the equation

   $$x^3 + y^3 = 2,$$

   at point $(1, 1)$.
   Solution: $\frac{dy}{dx} = -\frac{x^2}{y^2} = -1.$

   (b) $y(x)$ as a function of x defined by the equation

   $$xy^3 + x^2y^2 + x^3y = -1,$$

   at point $(-1, 1)$.
   Solution: $\frac{dy}{dx} = -\frac{y^3 + 2xy^2 + 3x^2y}{3xy^2 + 2x^2y + x^3} = 1.$

3. Consider the composition of three functions

   (a) Let $f(x)$, $g(x)$, and $h(x)$ be differentiable functions. Derive a formula for the derivative of the function $f(g(h(x)))$.
   Solution:
   $$\frac{d}{dx}\left(f(g(h(x)))\right) = f'(g(h(x)))g'(h(x))h'(x).$$

   (b) Write the function $\sqrt{1+\sqrt{1+x^2}}$ in the form $f(g(h(x)))$, and use your formula from above to compute the derivative of this function.
   Solution: Let $f(x) = \sqrt{1+x}$, $g(x) = \sqrt{1+x}$ and $h(x) = x^2$.
   $$\frac{d}{dx}\left(f(g(h(x)))\right) = \frac{x}{2\sqrt{1+x^2}\sqrt{1+\sqrt{1+x^2}}}.$$

4. Let $f(x) = (2x^2 + 3)^{3/2}$. Show that $f(x)$ is decreasing for $x < 0$ and increasing for $x > 0$. Sketch the graph of this function and find the global minimum of $f(x)$ for $-\infty < x < \infty$.
   Solution:
$$f'(x) = 6x\sqrt{2x^2+3}, \quad f''(x) = \frac{6(4x^2+3)}{\sqrt{2x^2+3}}.$$
   The factor $\sqrt{2x^2+3}$ in $f'(x)$ is always positive. Hence $f'(x) < 0$ is negative for $x < 0$ and positive for $x > 0$. Therefore $f(x)$ is decreasing for $x < 0$ and increasing for $x > 0$. Since $f''(x)$ is always positive, the graph is concave up, without inflection points. The minimum is at $x = 0$.

5. If the demand equation for a monopolist is $p = 150 - 0.02x$ and the cost function is $C(x) = 10x + 300$, find the value of x that maximizes the profit.
Solution: The profit

$$P(x) = xp - C(x) = x\,(150 - 0.02x) - (10x + 300) = 140x - 0.02x^2 - 300,$$

which is a quadratic with negative second order coefficient -0.02. Hence the graph is concave down, with max at

$$P'(x) = 140 - 0.04x = 0,$$

or $x = 3500$.

6. An open rectangular box is to be 4 feet long and have a volume of 200 cubic feet. Find the dimensions for which the amount of material needed to construct the box is as small as possible.
   Solution: Let the height be x feet and width y feet. Then the constraint is $4xy = 200$ or $xy = 50$. Since the box is open, one only needs to construct the four sides and the bottom of the box. Hence the objective to b e minimized is $4y + 2(xy + 4x) = 2xy + 4y + 8x$. Replace y by $50/x$, the objective is

$$C(x) = 2x * 50/x + 4 * 50/x + 8x = 100 + 200/x + 8x.$$

Letting

$$C'(x) = -200/x^2 + 8 = 0,$$

we have $x = 5$ and hence $y = 10$.