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# Math16A Sample Final Exam, Fall 2009

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**This is a closed book, closed notes exam. You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Look over the whole exam to find problems that you can do quickly. You need not simplify your answers unless you are specifically asked to do so.**

| Problem | Maximum Score | Your Score |
|---|---|---|
| 1 | 12.5 | |
| 2 | 12.5 | |
| 3 | 12.5 | |
| 4 | 12.5 | |
| 5 | 12.5 | |
| 6 | 12.5 | |
| 7 | 12.5 | |
| 8 | 12.5 | |
| Total | 100 | |

Your Name & SID: ______________________________

Your Section & GSI: ______________________________

1. (a) Compute the following indefinite integral:
$\int \left(x^2 + 1/x - e^{4x}\right) dx.$
**Solution:** $x^3/3 + \ln|x| - e^{4x}/4 + C.$

(b) Compute the following definite integral:
$\int_1^2 \left(\sqrt{2x+1} - x^{-2}\right) dx.$
**Solution:**

$$\int_1^2 \left(\sqrt{2x+1} - x^{-2}\right) dx = \left((2x+1)^{3/2}/3 + 1/x\right)_1^2 = \frac{5\sqrt{5} - 3\sqrt{3}}{3} - \frac{1}{2}.$$

2. (a) Suppose that the marginal revenue function for a company is 100 - x. Find the additional revenue received from doubling production if currently 10 units are being produced.

   **Solution:** Additional revenue is $\int_{10}^{20} (100 - x)\,dx = 850$.

   (b) Suppose that the marginal cost is $2x + 0.3x^2$, with fixed costs at 30. Find the total cost if 10 units are being produced.

   **Solution:** Actual cost is $\int (2x + 0.3x^2)\,dx = x^2 + 0.1x^3 + C$. Since fixed cost is 30, we have $C = 30$. Hence total cost for 10 units is 230.

3. (a) Compute the area of the region between the curves $y = x + 1$ and $y = -x^2 - 1$ from $x = 0$ to $x = 1$.
**Solution:** Area is
$$\int_0^1 \Big((x+1) - (-x^2 - 1)\Big)\, dx = \frac{17}{6}.$$
(b) Compute the area enclosed by the curves $y = 1/2x^2 + 1$ and $y = -1/2x^2 + 3x - 1$.
**Solution:** The curves intersect at $x = 1$ and $x = 2$. The area is
$$\int_1^2 \Big((-1/2x^2 + 3x - 1) - (1/2x^2 + 1)\Big)\, dx = \frac{1}{6}.$$

4. Suppose that a lake is stocked with 100 fish. After 1 month, there are 150 fish in the lake. An ecological study predicts that the lake can support 600 fish. Use a logistic growth curve to estimate the number of fish in the lake after 1 year.

   **Solution:** Let $f(t)$ be the fish population at end of $t$ months. Then there exist $k$ and $B$ such that

   $$f(t) = \frac{M}{1 + Be^{-Mkt}},$$

   where $M = 600$. By assumption, $f(0) = M/(1 + B) = 100$. It follows that $B = 5$. In addition, since $f(1) = 150$, we have

   $$600/\left(1 + Be^{-Mk}\right) = 150,$$

   it follows that $e^{-Mk} = 3/5$, and hence

   $$f(t) = \frac{600}{1 + 5\,(3/5)^t}.$$

   Let $t = 12$. The fish population at the end of the year is

   $$f(12) = \frac{600}{1 + 5\,(3/5)^{12}}.$$

5. (a) Find the average value of the function $f(x) = 1/x$ over the interval $-3 \leq x \leq -1$.
**Solution:** Average is $\frac{1}{(-1)-(-3)} \int_{-3}^{-1} 1/x dx = -1/2 \ln 3$.

(b) Use a Riemann sum to estimate the following sum for large enough values of $n$:

$$\sqrt{1+n} + \sqrt{2+n} + \sqrt{3+n} + \cdots + \sqrt{n+n}.$$

**Solution:** Define $x_j = 1 + jh$, where $h = 1/n$ and $j = 1, \cdots, n$. Then for each $j$, we have $\sqrt{j+n} = \sqrt{n}\sqrt{x_j}$. Sum is

$$\sqrt{n}\left(\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \cdots + \sqrt{x_n}\right).$$

Since $x_1 = 1 + h$ and $x_n = 2$, $h\left(\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \cdots + \sqrt{x_n}\right)$ is a Riemann sum approaching $\int_1^2 \sqrt{x} dx$, the sum is

$$\sqrt{n}\left(\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \cdots + \sqrt{x_n}\right) = n^{3/2} h \left(\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \cdots + \sqrt{x_n}\right) \approx n^{3/2} \int_1^2 \sqrt{x} dx.$$

6. If the demand equation for a monopolist is $p = 150 - 0.02x$ and the cost function is $C(x) = 10x + 300$, find the value of $x$ that maximizes the profit.
   **Solution:** The Profit is

   $$P(x) = x(150 - 0.02x) - 10x - 300,$$

   with $P'(x) = 140 - 0.04x$, and $P''(x) = -0.04 < 0$. Let $P'(x) = 0$, we obtain $x = 3500$. $P(3500)$ is the max since $P''(3500) < 0$.

7. (a) For $u = \sqrt{x}$ and $\dfrac{dy}{du} = u^2$, find $\dfrac{dy}{dx}$.
**Solution:**
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = u^2 \frac{d}{dx}\left(\sqrt{x}\right) = \sqrt{x}/2.$$

(b) For $g(x) = x^2$ and $f'(x) = x\sqrt{x+1}$, find a formula for $\dfrac{d}{dx}f(g(x))$.
**Solution:** Since $g'(x) = 2x$, by the chain rule
$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) = g(x)\sqrt{g(x)+1}(2x) = 2x^3\sqrt{x^2+1}.$$

8. Given function $f(x) = \ln(x^2 + 1)$.

   (a) Find the relative and absolute maximum/minimum of $f(x)$ if they exist.
   **Solution:** $f'(x) = 2x/(x^2 + 1)$ and $f''(x) = 2(1 - x^2)/(x^2 + 1)^2$. Let $f'(x) = 0$ we have $x = 0$. Since $f''(0) > 0$, $x = 0$ is relative minimum. Since $f(0) = 0 \leq f(x)$ for any $x$, $x = 0$ is also an absolute minimum.

   (b) Sketch the graph of this function.