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Math123 Ordinary Differential Equations: Sample Final Solutions

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This is an open book, open notes exam. You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Do as much as you can. Partial solutions will get partial credit. Look over the whole exam to find problems that you can do quickly. You need not simplify your answers unless you are specifically asked to do so.

Problem	Maximum Score	Your Score
1	20	
2	10	
3	20	
4	20	
5	10	
6	10	
Total	90	

Your Name: ______________________________

Your SID: ______________________________

1. (a) Suppose $\alpha > 0$ and let f be a continuous function on R with $|f(y)| \leq |y|^2$. Find an $\epsilon > 0$ such that every solution y of $y' = -\alpha y + f(y)$ with $|y(0)| < \epsilon$ exists for all $x \geq 0$.
Solution: This is an almost linear system. Let $y(x) = e^{-\alpha x} v(x)$, then the ODE simplifies to

$$v'(x) = e^{\alpha x} f\left(e^{-\alpha x} v(x)\right).$$

Integrating,

$$v(x) = v(0) + \int_0^x e^{\alpha \tau} f\left(e^{-\alpha \tau} v(\tau)\right) d\tau.$$

Since $v(0) = y_0$, we have

$$|v(x)| \leq |y_0| + \int_0^x e^{\alpha \tau} |f\left(e^{-\alpha \tau} v(\tau)\right)| d\tau \leq |y_0| + \int_0^x e^{\alpha \tau} \left(e^{-2\alpha \tau} |v(\tau)|^2\right) d\tau.$$

Assume for now that $|v(x)| \leq 1$ for all $x \geq 0$. Then

$$|v(x)| \leq |y_0| + \int_0^x e^{-\alpha \tau} |v(\tau)| d\tau.$$

By Gronwall inequality,

$$|v(x)| \leq |y_0| \mathbf{exp}\left(\int_0^x e^{-\alpha \tau} d\tau\right) = |y_0| \mathbf{exp}\left((1 - e^{-\alpha x})/\alpha\right) < |y_0| e^{1/\alpha}.$$

Now we choose $|y_0| \leq e^{-1/\alpha} < 1$. By existence and continuity of ODE solutions, we know $v(x)$ exists with $|v(x)| < 1$ for $0 \leq x \leq x_0$, where $x_0 > 0$ is sufficiently small. The above argument shows that in fact $|v(x)| < |y_0| e^{1/\alpha} < 1$ for all such x. By the extension technique in the book, this implies that in fact $v(x)$ exists for all $x > 0$ and $|v(x)| \leq |y_0| e^{1/\alpha} < 1$ for all x. So $\epsilon = e^{-1/\alpha}$. For any $|y_0| < \epsilon$, solution is bounded and hence exists for all time.

(b) By choosing the initial value $y(0)$, find a solution of $y' = -y + y^2$ that does not exist for all $x \geq 0$.
Solution: ODE has form $y' = y(y - 1)$, or

$$(1/(y - 1) - 1/y)\, dy = dt.$$

Integrating, ODE has solution

$$y(t) = \frac{y_0}{y_0 + (1 - y_0) e^t},$$

where $y(0) = y_0$. For any $y_0 > 1$, solution is undefined at $t = \ln\left(y_0/(y_0 - 1)\right)$.

2. Suppose that A is an $n \times n$ matrix whose eigenvalues all have nonzero real parts. Show that every solution y of $y' = Ay$ satisfies either $\|y(x)\| \to \infty$ or $\|y(x)\| \to 0$ as $x \to \infty$.
   Solution: If A is a Jordan block, then the solution either decays to zero or go to infinity depending on whether the eigenvalue has negative real part. In general, there exists a T such that
   $$A = T\mathbf{diag}(J_1, \cdots, J_r)T^{-1}$$
   where the J's are Jordan blocks. Let $y(t_0) = y_0$ be the initial condition so that the solution is
   $$y = T\mathbf{diag}(e^{J_1(t-t_0)}, \cdots, e^{J_r(t-t_0)})T^{-1}y_0 = T\begin{pmatrix} e^{J_1(t-t_0)}u_1 \\ \vdots \\ e^{J_r(t-t_0)}u_r \end{pmatrix}$$
   where $T^{-1}y_0 = \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix}$. Then each component $e^{J_k(t-t_0)}u_k$ will go to infinity if the eigenvalue value in J_k has positive real part and $u_k \neq 0$; it goes to zero otherwise. The solution y goes to zero if every component goes to zero, and go to infinity otherwise.

3. (a) Suppose that $f(x)$ is continuously differentiable satisfying $\|f(x) - f(y)\| \leq \|x - y\|$ for all x and y. Show that the solution of $y' = f(y)$ satisfies $\|y(x)\| \leq (e^x - 1)\|f(y_0)\| + \|y_0\|$ and exists for all $x > 0$.

Proof: Define $\phi_0(x) = y_0$ and

$$\phi_{k+1}(x) = y_0 + \int_0^x f(\phi_k(\tau))d\tau, \quad k = 0, 1, ...$$

Then for $x > 0$,

$$\|\phi_1(x) - \phi_0(x)\| \leq x\|f(y_0)\|.$$

In general,

$$\phi_{k+1}(x) - \phi_k(x) = \int_0^x \left(f(\phi_k(\tau)) - f(\phi_{k-1}(\tau))\right) d\tau, \quad k = 1, ...$$

and

$$\|\phi_{k+1}(x) - \phi_k(x)\| \leq \int_0^x \|\phi_k(\tau) - \phi_{k-1}(\tau)| \, d\tau, \quad k = 1, ...$$

A simple induction shows that

$$\|\phi_{k+1}(x) - \phi_k(x)\| \leq \frac{x^{k+1}}{(k+1)!}\|f(y_0)\|.$$

Thus, the sequence $\{\phi_k(x)\}_{k=0}^{\infty}$ converges and converges to the solution of the ODE, which can be written as

$$y(x) = \phi_0(x) + \sum_{k=0}^{\infty} \left(\phi_{k+1}(x) - \phi_k(x)\right),$$

and hence

$$\|y(x)\| = \|\phi_0(x)\| + \sum_{k=0}^{\infty} \|\phi_{k+1}(x) - \phi_k(x)\| \leq \|y_0\| + \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!}\|f(y_0)\| = (e^x - 1)\|f(y_0)\| + \|y_0\|.$$

(b) Let $y = \begin{pmatrix} \cos\left(\sqrt{10 - x}\right) \\ \sin\left(\sqrt{10 - x}\right) \end{pmatrix}$. Can y be the solution to a 2×2 autonomous system $y' = f(y)$ with a continuously differentiable function $f(y)$? If so, find such $f(y)$; otherwise, explain why not.

Solution: NO. Given y function is bounded, but undefined for $x > 10$. y' is also undefined at $x = 10$ and unbounded for x close to 10. If y was an ODE solution, it would have to be extendable to all $x > 0$ and have finite derivatives at $x = 10$.

4. Let $P = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Find

   (a) The matrix $Y(x) = e^{Px}$.
   Solution: $Y(x) = e^{2x} \begin{pmatrix} 1 & x \\ 01 & \end{pmatrix}$.

   (b) the solution of $y' = Py$ with $y(0) = (0, 1)^T$.
   Solution: Solution is $Y(x)(0, 1)^T = e^{2x} \begin{pmatrix} x \\ 1 \end{pmatrix}$.

5. Consider the ODE $y' = f(y)$, where $y = (y_1, y_2, y_3)^T$ and

$$f(y) = \begin{pmatrix} -2y_2y_3 \\ y_1y_3 + y_3^2y_2 \\ y_1y_2 - y_2^2y_3 \end{pmatrix}.$$

Define $V(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2$. Suppose $y = y(t)$ is a positive solution: $y_1(t) > 0$, $y_2(t) > 0$, $y_3(t) > 0$. Show that

$$V(y_1(t), y_2(t), y_3(t)) \leq V(y_1(0), y_2(0), y_3(0)).$$

Solution: We can show that

$$\frac{d}{dt}V(y_1(t), y_2(t), y_3(t)) = 0$$

Hence

$$V(y_1(t), y_2(t), y_3(t)) = V(y_1(0), y_2(0), y_3(0))$$

for all t.

6. Consider the equation $y' = f(y)$ with $y = (y_1, y_2)^T$, where

$$f(y) = \begin{pmatrix} y_2 - y_1 \\ -y_2(1 + y_1^2) \end{pmatrix}.$$

Show that $y = (0,0)^T$ is a critical point, and $y \equiv (0,0)^T$ is an asymptotically stable solution.
Solution:

$$f(y) = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} y + \begin{pmatrix} 0 \\ -y_2 y_1^2 \end{pmatrix}.$$

Hence ODE is almost linear. Theorem 4.3 will work. Alternatively, we can apply Theorem 5.5. with $V(y_1, y_2) = 1/2(y_2 - y_1)^2 + 1/4(1 + y_1^2)^2 - 1/4$. Show

- V is positive definite,
- $V^*(y_1, y_2) \leq 0$,
- $E = \{y | V^*(y) = 0\} = \{y | y_1 = y_2\}$,
- 0 is the only invariant subset of E