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Math123 Ordinary Differential Equations: Sample Final

This is an open book, open notes exam. You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Do as much as you can. Partial solutions will get partial credit. Look over the whole exam to find problems that you can do quickly. You need not simplify your answers unless you are specifically asked to do so.

Problem	Maximum Score	Your Score
1	20	
2	10	
3	20	
4	20	
5	10	
6	10	
Total	90	

Your Name: ______________________________

Your SID: ______________________________

1. (a) Suppose $\alpha > 0$ and let f be a continuous function on R with $|f(y)| \leq |y|^2$. Find an $\epsilon > 0$ such that every solution y of $y' = -\alpha y + f(y)$ with $|y(0)| < \epsilon$ exists for all $x \geq 0$.

   (b) By choosing the initial value $y(0)$, find a solution of $y' = -y + y^2$ that does not exist for all $x \geq 0$.

2. Suppose that A is an $n \times n$ matrix whose eigenvalues all have nonzero real parts. Show that every solution y of $y' = Ay$ satisfies either $\|y(x)\| \to \infty$ or $\|y(x)\| \to 0$ as $x \to \infty$.

3. (a) Suppose that $f(x)$ is continuously differentiable satisfying $\|f(x) - f(y)\| \le \|x - y\|$ for all x and y. Show that the solution of $y' = f(y)$ satisfies $\|y(x)\| \le (e^x - 1)\|f(y_0)\| + \|y_0\|$ and exists for all $x > 0$.

   (b) Let $y = \begin{pmatrix} \cos\left(\sqrt{10 - x}\right) \\ \sin\left(\sqrt{10 - x}\right) \end{pmatrix}$. Can y be the solution to a 2×2 autonomous system $y' = f(y)$ with a continuously differentiable function $f(y)$? If so, find such $f(y)$; otherwise, explain why not.

4. Let $P = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Find

   (a) The matrix $Y(x) = e^{Px}$.

   (b) the solution of $y' = Py$ with $y(0) = (0,1)^T$.

5. Consider the ODE $y' = f(y)$, where $y = (y_1, y_2, y_3)^T$ and

$$f(y) = \begin{pmatrix} -2y_2y_3 \\ y_1y_3 + y_3^2y_2 \\ y_1y_2 - y_2^2y_3 \end{pmatrix}.$$

   Define $V(y_1, y_2, y_3) = y_1^2 + y_2^2 + y_3^2$. Suppose $y = y(t)$ is a positive solution: $y_1(t) > 0$, $y_2(t) > 0$, $y_3(t) > 0$. Show that

$$V(y_1(t), y_2(t), y_3(t)) \le V(y_1(0), y_2(0), y_3(0)).$$

6. Consider the equation $y' = f(y)$ with $y = (y_1, y_2)^T$, where

$$f(y) = \begin{pmatrix} y_2 - y_1 \\ -y_2(1 + y_1^2) \end{pmatrix}.$$

Show that $y = (0,0)^T$ is a critical point, and $y \equiv (0,0)^T$ is an asymptotically stable solution.