MATH 54 - FINAL EXAM PRACTICE 3

1. Computations

Find the Fourier series for the function $f(x) = x^3 - 2x^2$, defined on $-\pi < x < \pi$:

Solution: First notice that here $L = \pi$. If we write f in a fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,$$

then the coefficients a_n and b_n are found by integrating, as follows:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} (x^3 - 2x^2)\cos nx\, dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \cos nx\, dx - \frac{1}{\pi}\int_{-\pi}^{\pi} 2x^2 \cos nx\, dx \\ &= 0 - \frac{1}{\pi}\int_{-\pi}^{\pi} 2x^2 \cos nx\, dx \quad (\text{ since } x^3 \cos nx \text{ is odd}) \end{aligned}$$

When $n = 0$, this is just

$$a_0 = -\frac{1}{\pi}\int_{-\pi}^{\pi} 2x^2\, dx = -\frac{1}{\pi}\left[\frac{2}{3}x^3\right]_{-\pi}^{\pi} = -4\pi^2/3.$$

For $n > 0$, this integral can be computed using the "tabular method" for repeated integration by parts, differentiating down the left column, and integrating down the right column, as follows:

$$\begin{array}{lcll} 2x^2 & & \cos nx & \\ & \searrow + & & \\ 4x & & \frac{1}{n}\sin nx & \longleftarrow (= 0 \text{ at } x = \pi \text{ and } -\pi) \\ & \searrow - & & \\ 4 & & -\frac{1}{n^2}\cos nx & \\ & \searrow + & & \\ 0 & & -\frac{1}{n^3}\sin nx & \longleftarrow (= 0 \text{ at } x = \pi \text{ and } -\pi) \end{array}$$

Thus the only term which survives in the integration by parts is (beware the many negative signs!):

$$a_n = -\frac{1}{\pi}\left[-4x\left(-\frac{1}{n^2}\cos nx\right)\right]_{-\pi}^{\pi} = -2\frac{1}{\pi}\left[4x\left(\frac{1}{n^2}\cos nx\right)\right]_0^{\pi} = -8\frac{1}{n^2}(-1)^n = \frac{8}{n^2}(-1)^{n+1}.$$

Next we compute the b_n's:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x^3-2x^2)\sin nx\,dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}x^3\sin nx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi}2x^2\sin nx\,dx\\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}x^3\sin nx\,dx \quad (\text{ since } 2x^2\sin nx \text{ is odd})
\end{aligned}$$

To evaluate this integral, we again use the tabular method:

$$\begin{array}{lcl}
x^3 & & \sin nx \\
 & \searrow + & \\
3x^2 & & -\frac{1}{n}\cos nx \\
 & \searrow - & \\
6x & & -\frac{1}{n^2}\sin nx \longleftarrow (= 0 \text{ at } x = \pi \text{ and } -\pi) \\
 & \searrow + & \\
6 & & \frac{1}{n^3}\cos nx \\
 & \searrow - & \\
0 & & \frac{1}{n^4}\sin nx \longleftarrow (= 0 \text{ at } x = \pi \text{ and } -\pi)
\end{array}$$

So the integral is

$$\begin{aligned}
b_n &= \frac{1}{\pi}\left[-x^3\frac{1}{n}\cos nx + 6x\frac{1}{n^3}\cos nx\right]_{-\pi}^{\pi}\\
&= 2\frac{1}{\pi}\left[-x^3\frac{1}{n}\cos nx + 6x\frac{1}{n^3}\cos nx\right]_{0}^{\pi}\\
&= -2\pi^2\frac{1}{n}\cos n\pi + 12\frac{1}{n^3}\cos n\pi\\
&= -2\pi^2\frac{1}{n}\cos n\pi + 12\frac{1}{n^3}\cos n\pi\\
&= -2\pi^2\frac{1}{n}(-1)^n + 12\frac{1}{n^3}(-1)^n\\
&= (-1)^n\frac{2}{n}\left[\frac{6}{n^2}-\pi^2\right]
\end{aligned}$$

Putting this all together, we obtain the following fourier series expansion for f:

$$f(x) \sim \frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\frac{8}{n^2}(-1)^{n+1}\cos nx + \sum_{n=1}^{\infty}(-1)^n\frac{2}{n}\left[\frac{6}{n^2}-\pi^2\right]\sin nx$$

2. Linear Algebra

(1) In a vector space V, which of the following statements is false, for vectors $\mathbf{u}$ and $\mathbf{v}$ in V, and scalars c and d?
(a) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$
(b) $c(d\mathbf{u}) = (cd)\mathbf{u}$
(c) $\mathbf{u}(c\mathbf{u}+\mathbf{v}) = c\mathbf{u}+\mathbf{uv}$
(d) $(c+d)(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+d\mathbf{v}+c\mathbf{v}+d\mathbf{u}$

Solution:
(c) is false. For one thing, $\mathbf{uv}$ doesn't even make sense, since in a general vector space, we usually can't multiply vectors, only multiply a vector by a scalar. Furthermore, even if we could, the term involving c on the right seems to be missing a factor of $\mathbf{u}$.

(2) Which of the following is a vector space?
(a) The set of all polynomials $p(x)$ such that $p(1) \neq 1$.
(b) The set of all 3×3 matrices of determinant one.
(c) The set of points $\{(x, y) \,|\, x = y \text{ or } x = -y\}$ in $\mathbb{R}^2$
(d) The set of all sequences of real numbers.

Solution:
(a) is not a vector space, since if we have two polynomials p_1 and p_2, for which $p_1(1) \neq 1$ and $p_2(1) \neq 1$, it may still be true that $p_1 + p_2$ is equal to 1 when evaluated at 1. For instance, take $p_1(x) = 1/2$ and $p_2(x) = 1/2$, both constant polynomials.
(b) is not a vector space, since the only matrix which could play the role of "zero vector" would be the zero matrix, but this one does not have determinant one.
(c) is not a vector space since it is not closed under addition in $\mathbb{R}^2$: the vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are both in this set, but their sum is the vector $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, which is not in the set. [Note, though, that this set *is* closed under scalar multiplication]
(d) is a vector space. The sum of two sequences $\{a_n\}$ and $\{b_n\}$ is a new sequence $\{a_n + b + n\}$, and for any real number c and sequence $\{a_n\}$, we get a new sequence $\{ca_n\}$. It is tedious, but not too hard to check that all the properties of addition and scalar multiplication hold.

(3) Which of the following is a *sub*space of $\mathbb{R}^2$?
(a) The set of all points (a, b) such that $f(a, b) = 0$, where f is the function $f(x, y) = x^2 + y^2$
(b) The set of eigenvectors for the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$
(c) The set of all $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x + 2y + z = 0$.
(d) South Dakota.

Solution:
(a) consists only of the single point $(0, 0)$. This is the zero subspace of $\mathbb{R}^2$. It is the correct answer.
(b) is not a subspace since the zero vector is not an eigenvector - so this set does not contain the zero vector [note that when we speak of an "eigenspace" we mean a set of eigenvectors plus the zero vector. We have to throw in the zero vector in addition, to say that it is actually a vector space]
(c) is a subspace of $\mathbb{R}^3$, but not of $\mathbb{R}^2$ - the elements of this set aren't even IN $\mathbb{R}^2$!
(d) Alas, South Dakota, too, is three-dimensional (or four? or eleven? - O physics, tell us what dimension it really is!). Thus it cannot be in $\mathbb{R}^2$, as a subspace or otherwise.