Math 113, Summer 2014: **Solution to Exam 1**

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Thursday, July 17th

Attempt to answer all of the following FIVE questions. You DO NOT need to justify your response to the TRUE/FALSE problems.

1. This is a closed book exam. Please put away all your notes, textbooks, calculators and portable electronic devices and turn your mobile phones to 'silent' (non-vibrate) mode.

2. Explain your answers **CLEARLY** and **NEATLY**, and in **COMPLETE ENGLISH SENTENCES**. State all theorems you have used from class. To receive full credit you will need to justify each of your calculations and deductions coherently and fully.

3. Correct answers without appropriate justification will be treated with great skepticism.

4. Write your name on this exam and any extra sheets you hand in.

Question 1:	/20
Question 2:	/20
Question 3:	/25
Question 4:	/30
Question 5:	/30
Total:	/125

Name: ____________________________________

1. (20 points)

   (a) Let H be a subgroup of a group G. Give the definition of a left-H coset.

   (b) Prove that if $g \in G$, $gH = H$ if and only if $g \in H$.

   (c) If $G = S_3$, find a subgroup $H \subseteq S_3$, and an element $g \in S_3$ such that $gH \neq Hg$.

   Solution:

   (a) A left H-coset is a set $\{gh \,|\, h \in H\}$. Equivalently, it is an equivalence class of the equivalence relation $g \sim g'$ if and only if $g^{-1}g'\mathfrak{n}H$. Both answers were accepted.

   (b) First assume that $gH = H$. Then since $g = ge \in gH = H$, $g \in H$. Conversely, suppose $g \in H$. Then $g \sim e$ under the above equivalence relation. We proved in class that equivalent elements represent the same equivalence class, so $gH = eH = H$.

   (c) Take H to be $< (12) >$, the subgroup generated by (12); pick $g = (123)$. Then $gH = \{(123), (13)\}$, but $Hg = \{(123), (23)\}$.

2. True/False (20 points - 2 points each). No justification required.

   (a) There exists a nonabelian group G such that every subgroup of G is normal in G. **True:** the group $\mathcal{Q}$ of quaternions is such a group.

   (b) The permutations (1435)(23456) and (123)(456) are conjugate in S_6. **False**: We have $(1435)(23456) = (14)(256)$, so they don't have the same cycle type, hence are not conjugate.

   (c) The center $Z(G)$ of a group G can be written $Z(G) = \bigcup_{g \in G} \mathrm{Cent}_G(g)$. **False**: The center is the *intersection* of the centralizers.

   (d) A subgroup H of G is normal in G if and only if H can be written as a union of conjugacy classes. **True**: both conditions ensure that all conjugates of elements of H lie in H.

   (e) For each $1 \leq m \leq n$, there is a subgroup of S_n isomorphic to $\mathbb{Z}/m\mathbb{Z}$. **True**: The subgroup generated by any m-cycle has order m and is cyclic, hence is isomorphic to $\mathbb{Z}/m\mathbb{Z}$.

   (f) There exists a surjective homomorphism $D_{10} \to \mathbb{Z}/2\mathbb{Z}$. **True**: one such homomorphism is defined by sending r to $\overline{0}$ and s to $\overline{1}$.

   (g) If a finite group G acts on a set X in such a way that there is just one orbit for this action, then $|X| \leq |G|$. **True**: The orbit stabilizer theorem says that the size of this orbit $\mathcal{O}$ (which is equal to X), is $|G|/\mathrm{Stab}_G(x)$ (for any $x \in X$), which is at most $|G|$.

   (h) In S_4, the elements of order 3, together with the identity element, form a subgroup. **False**: $(123)(234) = (12)(34)$, so this set is not closed under composition.

   (i) If G acts on the set X, and $x \in X$, then $\mathrm{Stab}_G(x)$ is normal if and only if x is a fixed point of the action. **False**: x is a fixed point if and only the stabilizer x is *all* of G, which is a normal subgroup. But if the stabilizer is normal, it still may not be all of G, and x would not be a fixed point. For example, when N is a proper normal subgroup of G, then G acts on G/N by permutation. The stabilizer of the coset N is N itself, which is normal. But N is not a fixed point.

   (j) There is an action of $\mathbb{Z}/3\mathbb{Z}$ on $\mathbb{Z}/p^3\mathbb{Z}$ given by the formula $\overline{i} \cdot \overline{j} = \overline{p^i j}$. **False**: This expression is not well-defined. For example, $\overline{1} \cdot \overline{j} = \overline{pj}$, which is not necessarily $\overline{0}$; but we could also write $\overline{1} = \overline{4}$, and get $\overline{4} \cdot \overline{j} = \overline{p^4 j} = \overline{p^3 pj} = \overline{0}$.

3. (25 points)

   (a) Prove that $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/49\mathbb{Z}$.
   (b) Let A be an abelian group of order 392. List all possible isomorphism classes of A.
   (c) Assume further that A contains an element of order 196. List the possible isomorphism classes of A.
   (d) Let $G = \mathbb{Z}/49\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Find subgroups $H, K \subset G$ of order 2 such that G/H and G/K are not isomorphic.

Solution:

(a) $\mathbb{Z}/49\mathbb{Z}$ contains an element of order 49, but $\mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$ does not, because for any $(\overline{a}, \overline{b}) \in \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}$, $7(\overline{a}, \overline{b}) = (\overline{7a}, \overline{7b}) = (\overline{0}, \overline{0})$, so the order of $(\overline{a}, \overline{b})$ is at most 7.

(b) $392 = 2^3 \cdot 7^2$. The isomorphism classes are

$$\begin{aligned}
&\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/49\mathbb{Z} \\
&\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/49\mathbb{Z} \\
&\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/49\mathbb{Z} \\
&\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \\
&\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \\
&\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/7\mathbb{Z}
\end{aligned}$$

(c) An element of order $196 = 2^2 \cdot 7^2$ must be a product of an element of order 4 and an element of order 49, so out of the above list we select those groups containing elements of both order 4 and 49, namely

$$\begin{aligned}
&\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/49\mathbb{Z} \\
&\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/49\mathbb{Z}
\end{aligned}$$

(d) We will sloppily write just 0 for a trivial group. The subgroups $H = 0 \times < \overline{2} > \times 0$ and $K = 0 \times 0 \times \mathbb{Z}/2\mathbb{Z}$ both have order two. But $G/H \cong \mathbb{Z}/49\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ whereas $G/K \cong \mathbb{Z}/49\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \times 0$. These are not isomorphic by an analogous argument to that in part (a).

4. (30 points)

   (a) Give the definition of a Sylow p-subgroup of a finite group G.

   (b) Let G be a group of order 99. Prove that there is exactly one Sylow 3-subgroup H and exactly one Sylow 11-subgroup K.

   (c) If G is a group of order $p^r m$ with $\gcd(p, m) = 1$, $m > 1$, and $|G| > m!$, prove that G is not simple[1]. [Hint: find a certain subgroup, and let G act on its left cosets]

Solution:

   (a) Let p be a prime dividing $|G|$, with p^r being the largest power of p which divides $|G|$. Then a **Sylow p-subgroup** of G is a subgroup of order p^r. Equivalently, it is a *maximal* subgroup whose order is a power of p, but this was not the definition we gave in class.

   (b) First factor $|G| = 99 = 3^2 \cdot 11$. Using the notation from class, we must show that $k_3 = k_{11} = 1$. By SYL3 and one of the corollaries we have $k_3 \equiv 1 \mod 3$, so $k_3 = 1, 4, 7, 10, ...$ and $k_3|11$. So $k_3 = 1$. Similarly, $k_{11} \equiv 1 \mod 11$, so $k_{11} = 1, 12, 23, ...$; but $k_{11}|9$, so $k_3 = 1$.

   (c) By SYL1, there is at least one Sylow p-subgroup S. Its order is p^r, so its index is $[G : S] = m$; this is the size of G/S (which may not be a group, since we do not know whether S is normal). We let G act on this set G/S of left S-cosets by permutation: $g \cdot g'S = gg'S$. This induces a homomorphism $f\colon G \to \mathrm{Perm}(G/S) \cong S_m$. The first isomorphism theorem tells us that

$$\mathrm{im} f \cong G/\ker f,$$

   so that $|\ker f| = |G|/|\mathrm{im} f|$. But $|\mathrm{im} f| \leq |S_m| = m!$, since $\mathrm{im} f$ is a subgroup of S_m, and we are given that $|G| > m!$, so $|G|/|\mathrm{im} f| > 1$, which implies the kernel is nontrivial. Also, the kernel is not all of G, because this would mean that the action is trivial, but for $g \notin S$ (there is such a g, since the index of S is $m > 1$), $gS \neq S$, so the action is not trivial. Finally, we have seen in class that the kernel of any map is normal, so $\ker f$ is a proper nontrivial normal subgroup of G, so G is not simple.

[1] Recall that a group is **simple** if it has no proper nontrivial normal subgroups.

5. (30 points)

   (a) Let $K = \{e, (12)(34), (13)(24), (14)(23)\} \subset S_4$. The subgroup $< (23) >$ acts on K by conjugation. How many orbits are there for this action?

   Now let H and K be normal subgroups of an arbitrary finite group G.

   (b) The subgroup H acts on K by conjugation. If $H \cap K = \{e\}$, prove that this action is trivial.

   (c) Now let H act instead on G/K by left translation via $h \cdot (gK) = (hg)K$. If $|G| = |H||K|$ and there is only one orbit of this action, prove that $H \cap K = \{e\}$.

   (d) In the situation of part (c), prove that the map $\phi \colon H \times K \to G$, $\phi(h, k) = hk$, is an isomorphism.

Solution:

(a) The orbits are $\mathcal{O}_e = \{e\}$, $\mathcal{O}_{(12)(34)} = \{(12)(34), (13)(24)\}$, and $\mathcal{O}_{(14)(23)} = \{(14)(23)\}$.

(b) First of all, the action makes sense, since if $h \in H$, $k \in K$, then $h \cdot k = hkh^{-1} \in K$ because K is normal. To show the action is trivial, pick any $h \in H$ and $k \in K$. We wish to show that $h \cdot k = k$. This is equivalent to showing that $hkh^{-1}k^{-1}$=e. Since $hkh^{-1}k^{-1} = (hkh^{-1}k^{-1}$, and $hkh^{-1} \in K$, we get $hkh^{-1}k^{-1} \in K$ since K is closed under multiplication. Similarly, since $hkh^{-1}k^{-1} = h(kh^{-1}k^{-1})$, and $kh^{-1}k^{-1} \in H$ (as H is normal), we get $hkh^{-1}k^{-1} \in H$. Thus $hkh^{-1}k^{-1} \in H \cap K$, which is trivial, so $hkh^{-1}k^{-1} = e$, as desired.

(c) We denote the one orbit by $\mathcal{O} = \mathcal{O}_K$, which is equal to all of G/K. By the orbit-stabilizer theorem, we get $|\mathcal{O}| = |H|/|\mathrm{Stab}_G(K)|$. Replacing $|\mathcal{O}| = |G/K| = |G|/|K|$, and using $|G| = |H||K|$, this gives $|\mathcal{O}| = |H||K|/|K| = |H|$. Thus the orbit-stabilizer theorem tells us that $|\mathrm{Stab}_G(K)| = 1$, so the stabiliser is trivial. Thus we will be done if we can show that $H \cap K$ is equal to the stabilizer of K under this action.

For this, note that an element $h \in H$ stabilizes K if and only if $hK = K$, which happens if and only if $h \in K$. So the stabilizer consists precisely of those elements in $H \cap K$. So $H \cap K = \mathrm{Stab}_H(K) = \{e\}$.