1/27

- Office Hours: Thu 1130 - 130
  Fri 11 - 12

Note change

- Announce Piazza again
- HW Due Friday

3 Useful / Important Points About Proof-writing

① Really important! Do NOT start from the thing you're trying to prove, then deduce something else that's true

Example of a <u>bad</u> proof:

<u>Theorem</u>: All humans are female.

"Proof": All humans are female. (thing we want to show)

All females have an X-chromosome (fact from biology)

Therefore all humans have an X-chromosome.

This is a true fact of biology, therefore our initial claim is true. //

This is obviously wrong - The problem w/ the logic is we started w/ something false and argued something true - you can always do this! <u>A real proof starts</u> w/ something known to be true, and then <u>deduces something</u>

For a more mathematical, but still bogus proof:

Theorem: $0=1$

"Proof" $0=1$ (what we want to show)

$0\cdot 0 = 0 \cdot 0$ (mult. both sides by 0.

$0=0$ (by Prop last week)

Since $0=0$ is true, so is $0=1$. //

It doesn't work: to go backwards, we have to divide by zero!

(2) "If-and-only-if" proofs.

"P iff Q" means "if P then Q and if Q then P"

Usually, you have to prove both directions separately. Often one way is much easier

Example

Prop: Let $(v_1, \dots, v_n)$ be an independent list of vectors in V, all distinct, and let $U_1 = \mathrm{Span}\{v_1\}, \dots, U_n = \mathrm{Span}\{v_n\}$. Then

$V = U_1 + \dots + U_n$ iff $V = U_1 \oplus \dots \oplus U_n$

Proof "$\Leftarrow$" [This direction is "if $V = U_1 \oplus \dots \oplus U_n$, then $V = U_1 + \dots + U_n$]

We have to show that $V = U_1 + \dots + U_n$.

But this is part of the definition of direct sum. [Notice that this direction is very easy.. We didn't even use the linear independence!]

notes, not part of proof.

"$\Rightarrow$" [We use Prop 1.8. which says we have to check

(i) $V = U_1 + \dots + U_n$

(ii) The only way to write 0 as a sum $u_1 + \dots u_n$, $u_i \in U_i$, is with $u_i = 0$ for all i]

(i) $U_1 + \dots + U_n = V$, by our assumption

(ii) Let $0 = u_1 + \dots + u_n$, where each $u_i \in U_i$

We have to check $u_i = 0$ for all i.

We have

Since each $u_i \in U_i$, and $U_i = \mathrm{Span}\{v_i\}$,

we have $u_i = c_i v_i$ for some $c_i \in \mathbb{F}$.

So $0 = c_1 v_1 + \dots + c_n v_n$.

Since $(v_1, \dots, v_n)$ is independent, $c_i = 0$ $\forall i$.

Thus $u_i = 0 \cdot v_i = 0$ $\forall i$.

Therefore by Prop 1.8, $V = U_1 \oplus \dots \oplus U_n$.

This proves the "$\Leftarrow$" direction. //

(3) To prove 2 sets are equal, say $A = B$, you must show $A \subseteq B$ <u>and</u> $B \subseteq A$.

Again, it is often the case that one inclusion is much easier than the other.

Example:

<u>Prop:</u> $\mathbb{R} = \{ z \in \mathbb{C} \mid z = \bar{z} \}$ ($\bar{z}$ is the complex conjugate: $\overline{a+bi} = a - bi$)

<u>Proof</u> 1) $\mathbb{R} \subseteq \{ z \in \mathbb{C} \mid z = \bar{z} \}$

<u>Proof:</u> Let $a = a + 0i \in \mathbb{R}$

then $\bar{a} = \overline{a + 0i} = a - 0i = a$ ✓

so $a \in \{ z \mid z = \bar{z} \}$

2) $\{ z \mid z = \bar{z} \} \subseteq \mathbb{R}$

<u>Proof:</u> Let $z \in \mathbb{C}$ be such that $z = \bar{z}$

Then if ~~as~~ $z = a + bi$, we have

$$z = \bar{z} \Rightarrow a + bi = a - bi$$

$$\Rightarrow 2bi = 0$$

$$\Rightarrow b = 0$$

$$\Rightarrow z = a + 0i \text{ so } z \in \mathbb{R} \quad //.$$

Rest of Class: WS.

trick question

T/F Answers: (1) F (2) F (3) F (4) T (5) F (6) T (7) F (8) F (9) F (10) T