Bad Arguments

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I have written some arguments for various claims in HW2, # 7. Each contains (at least) one error, either logical, stylistic, or perhaps a missing explanation of some point. Read these carefully and mark where you think the errors are. Try to pin down exactly what's wrong in each case. Discuss them with your neighbors.

1. **Claim:** If $(v_1, \dots, v_{n-1})$ is linearly independent, and $v_n \notin \text{Span}(v_1, \dots, v_{n-1})$, then $(v_1, \dots, v_n)$ is linearly independent.

   Proof. If $(v_1, \dots, v_n)$ were linearly dependent, then by the linear dependence lemma 2.4, v_n would be in the span of $(v_1, \dots, v_{n-1})$, contradicting our choice of v_n. □

2. **Claim:**If V is infinite-dimensional, then there exists a sequence $v_1, v_2, \dots$ such that $(v_1, \dots, v_n)$ is linearly independent, for all n.

   Proof 1. Assume V is infinite-dimensional. The list (v_1) is linearly independent, because $v_1 \neq 0$ (V cannot be the zero space, since it's infinite-dimensional). For every $n > 1$, the list $(v_1, \dots, v_n)$ is independent because v_n is not in the span of $(v_1, \dots, v_{n-1})$ (if it were, V would have been finite-dimensional, contradiction). Thus we have produced an infinite sequence of v_i's that are all linearly independent from each other. □

   Proof 2. We prove the contrapositive: if there exists a sequence $v_1, v_2, \dots$ such that $(v_1, \dots, v_n)$ is **not** linearly independent for all n, then V is finite-dimensional. To show this, let $v_1, v_2, \dots$ be the given sequence. Since $(v_1, \dots, v_n)$ is not linearly independent for all n, there is some k such that the list $(v_1, \dots, v_k)$ is dependent. Let's choose k to be the smallest positive integer such that $(v_1, \dots, v_k)$ is dependent. Then by removing the kth vector, we get an independent list $(v_1, \dots, v_{k-1})$. We know from Theorem 2.12 that every linearly independent list can be extended to a basis. Since $(v_1, \dots, v_{k-1})$ is an independent list which, whenever a vector v_k is added, yields a dependent list, it must already be a basis. Therefore V has dimension k. In particular, it is finite-dimensional. □

3. **Claim:** If there exists a sequence $v_1, v_2, \dots$ such that $(v_1, \dots, v_n)$ is linearly independent, for all n, then V is infinite-dimensional.

   Proof. Suppose for contradiction that V is finite-dimensional. Then by definition of finite-dimensional, there is a list that spans V. Let this list be $(v_1, \dots, v_n)$. By our assumption, the list $(v_1, \dots, v_n, v_{n+1})$ is linearly independent. But this independent list is longer than our spanning list $(v_1, \dots, v_n)$, and theorem 2.6 says that any spanning list is at least as long as any independent list. So we have arrived at a contradiction, and hence V is infinite-dimensional. □

4. **Claim:** All math GSIs are hedgehogs.

   Proof. The statement is false! We give a counterexample: James is not a hedgehog. □