

Mathematics 212 – Complex Analysis in Several Variables

Fall 2007 – M. Christ

A point thoroughly botched in class

Let δ be one of the distance functions discussed: $\delta : \mathbb{C}^d \rightarrow [0, \infty)$ is continuous, $\delta(z) = 0$ if and only if $z = 0$, and $\delta(\lambda z) = |\lambda|\delta(z)$ for all scalars $\lambda \in \mathbb{C}^1$. We have for sets A, B the associated distance $\delta(A, B) = \inf_{(a,b) \in A \times B} \delta(a - b)$. (It is not claimed that δ satisfies any triangle inequality.)

To any nonzero $w \in \mathbb{C}^d$ is associated a degenerate norm function $\tilde{\delta}_w(z)$, defined to be $\delta(z)$ if z is a scalar multiple of w , and $+\infty$ otherwise. $\tilde{\delta}_w$ is the distance function associated to a degenerate rotated polydisk which has $d - 1$ polyradii equal to zero, and a single nonzero radius oriented in the direction of w . $\tilde{\delta}_w$ gives rise to a distance function on sets, also denoted $\tilde{\delta}_w$, in the same way that δ does. In what follows, we always assume implicitly that $w \neq 0$.

By the definitions, $\delta(z) = \inf_{w \neq 0} \tilde{\delta}_w(z)$ for all z . Moreover,

$$\delta(A, B) = \inf_w \tilde{\delta}_w(A, B)$$

for any sets A, B . Indeed, since two infima can be interchanged,

$$\begin{aligned} \delta(A, B) &= \inf_{(a,b) \in A \times B} \delta(a - b) = \inf_{(a,b) \in A \times B} \inf_w \tilde{\delta}_w(a - b) \\ &= \inf_w \inf_{(a,b) \in A \times B} \tilde{\delta}_w(a - b) = \inf_w \tilde{\delta}_w(A, B). \end{aligned}$$

We claim that for any nonzero $w \in \mathbb{C}^d$, for any domain of holomorphy Ω and any compact subset K ,

$$\tilde{\delta}_w(\widehat{K}, \Omega) = \tilde{\delta}_w(K, \Omega). \tag{1}$$

(I write \widehat{K} as shorthand for \widehat{K}_Ω .) Then

$$\delta(\widehat{K}, \partial\Omega) = \inf_w \tilde{\delta}_w(\widehat{K}, \partial\Omega) = \inf_w \tilde{\delta}_w(K, \partial\Omega) = \delta(K, \partial\Omega),$$

completing the proof.

(1) is a limiting version of what was proved correctly in class for distance functions associated to polydisks. We deduce it from the nondegenerate case, by approximating by genuine polydisks. By rotating coordinates we may assume that $w = (t, 0, \dots, 0)$ for some $t \in \mathbb{C}^1$. In these coordinates, define the polydisks $D_\varepsilon = \{z : |z_1| < |t| + \varepsilon \text{ and } |z_j| < \varepsilon \text{ for all } j > 1\}$. Then $\tilde{\delta}_w$ is associated to the degenerate polydisk $\{z : |z_1| < |t| \text{ and } z_j = 0 \text{ for all } j > 1\}$.

As ε decreases, the polydisks D_ε decrease, and therefore the associated distance functions δ_{D_ε} increase. Because D_ε contains the degenerate polydisk in question, $\delta_{D_\varepsilon} \leq \tilde{\delta}$. Moreover, because $\partial\Omega$ is closed, $\delta_{D_\varepsilon}(\zeta, \partial\Omega) \rightarrow \tilde{\delta}_w(\zeta, \partial\Omega)$ for any ζ .

We know already that $\delta_{D_\varepsilon}(z, \partial\Omega) \geq \delta_{D_\varepsilon}(K, \partial\Omega)$ for all $\varepsilon > 0$, and merely need to convert this to information on $\tilde{\delta}_w$. We claim that

$$\lim_{\varepsilon \rightarrow 0} \delta_{D_\varepsilon}(K, \partial\Omega) = \tilde{\delta}_w(K, \partial\Omega). \quad (2)$$

This requires some proof, since the definition of any distance between two sets involves an infimum which must be interchanged with the limit; this limit is a *supremum*, not an infimum.

If (2) were false, then the left-hand side would be strictly smaller, so there must exist a sequence $\varepsilon_j \rightarrow 0$, points $z_j \in K$, $\zeta_j \in \partial\Omega$, and ρ strictly less than $\tilde{\delta}_w(K, \partial\Omega)$ such that $z_j - \zeta_j \in \rho D_{\varepsilon_j}$ for all j . This forces the points ζ_j all to lie in some common bounded subset of \mathbb{C}^d . Therefore there exists a subsequence $j_\nu \rightarrow \infty$ such that $z_{j_\nu} \rightarrow z^*$ and $\zeta_{j_\nu} \rightarrow \zeta^*$ for some points $z^* \in K$ and $\zeta^* \in \partial\Omega$. Passing to the limit in the relation $z_j - \zeta_j \in \rho D_{\varepsilon_j}$ forces $z^* - \zeta^* = \lambda w$ for some scalar $\lambda \in \mathbb{C}^1$ with $|\lambda| \leq \rho$. But this means that $\tilde{\delta}_w(K, \partial\Omega) \leq \rho$, a contradiction. \square

Perhaps it would be better to prove (1) directly (via a modification of the proof of its nondegenerate sibling identity), rather than to deduce it from the nondegenerate version. That's not a bad exercise for our next problem set.