Mathematics 105 — Spring 2004 — M. Christ
Problem Set 9 — Solutions to Selecta

IX.A Consider the measure space $(\mathbb{R}^1, \overline{\mathcal{B}}_{\mathbb{R}^1}, \lambda)$ where λ denotes Lebesgue measure. Consider the measurable functions $f_n(x) = \frac{1}{n}\chi_{[0,n]}$. Show that $f_n \to 0$ uniformly on $\mathbb{R}$. Show that $\int f_n \, d\lambda \to 1$. Explain why this does not contradict any of our three basic convergence theorems.

Solution. That $f_n \to 0$ uniformly is obvious: Given $\varepsilon > 0$, choose an integer $N > 1/\varepsilon$. Then whenever $n \geq N$, for any $x \in \mathbb{R}$, either $|f_n(x) - 0| = n^{-1} < \varepsilon$, or $|f_n(x) - 0| = |0 - 0| < \varepsilon$. Since N does not depend on x, this establishes uniform convergence. It's equally obvious that $\int f_n \, d\lambda = n^{-1}\lambda([0,n]) = n/n = 1$. On the other hand, $\int \lim_{n\to\infty} f_n \, d\lambda = \int 0 \, d\lambda = 0$, which is not equal to $1 = \lim_{n\to\infty} \int f_n \, d\lambda$.

This doesn't contradict the monotone convergence theorem, because the hypothesis that $f_n \leq f_{n+1}$ for all n is violated.

There's no violation of the dominated convergence theorem. If $f_n(x) \leq g(x)$ for all x, n, then of course $g(x) \geq \sup_n f_n(x)$, and if $k - 1 < x < k$ for some positive integer k then $\sup_n f_n(x) = k^{-1}$. Thus $g(x) \geq k^{-1}$ for all $x \in (k-1, k)$. Therefore for any "Lebesgue dominator" g, $\int g \, d\lambda \geq \sum_{k=1}^{\infty} k^{-1}\lambda((k-1,k)) = \sum_{k=1}^{\infty} k^{-1} = +\infty$. Therefore there is no *integrable* Lebesgue dominator, so the Dominated Convergence Theorem doesn't apply.

The hypotheses of Fatou's lemma *are* satisfied — but there's no contradiction since Fatou's lemma doesn't assert equality, only the inequality $\int 0 \, d\lambda \leq \lim_{n\to\infty} 1 = 1$. □

IX.B Let $(E, \mathcal{A}, \mu)$ be any measure space. Suppose that $f : E \to \mathbb{R}^1$ is a measurable function (which never takes on the values $\pm\infty$). Recall that for any *Borel* measurable set $S \subset \mathbb{R}^1$, the set $f^{-1}(S)$ belongs to $\mathcal{A}$. (You need not prove this.). Define $\nu(S) = \mu(f^{-1}(S))$ for all $S \in \mathcal{B}_{\mathbb{R}^1}$ Show that ν is a measure on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$.

More generally, suppose that $f : E \to \mathbb{R}^n$ is a measurable mapping, in the sense that $f^{-1}(S) \in \mathcal{A}$ for all $S \in \mathcal{B}_{\mathbb{R}^n}$. Show that the above recipe defines a measure on $\mathcal{B}_{\mathbb{R}^n}$.

Solution. $\nu(A)$ is defined for all Borel measurable subsets $A \subset \mathbb{R}^n$, and $\nu(A) \geq 0$. $\nu(\emptyset) = 0$ since $f^{-1}(\emptyset) = \emptyset$. It remains only to show that if $\{A_j : 1 \leq j < \infty\}$ is a countable family of pairwise disjoint Borel subsets of $\mathbb{R}^n$, then $\nu(\cup_j A_j) = \sum_j \nu(A_j)$. Now $f^{-1}(\cup_j A_j) = \cup_j f^{-1}(A_j)$, and the sets $f^{-1}(A_j)$ are pairwise disjoint if the sets A_j are. Thus $\nu(\cup_j A_j) = \mu(f^{-1}(\cup_j A_j)) = \mu(\cup_j f^{-1}(A_j)) = \sum_j \mu(f^{-1}(A_j)) = \sum_j \nu(A_j)$. □

IX.C Recall the fat, or generalized, Cantor set $\mathcal{C}_\alpha$ defined by starting with $[0,1]$, deleting a subinterval centered at its midpoint of length $\alpha/3$, then deleting from each of the resulting two intervals centered subintervals of lengths $\alpha/3^3$, and so forth. Show directly (that is, using properties of $\mathcal{C}_\alpha$ but not using the theorem) that the characteristic function of $\mathcal{C}_\alpha$ satisfies the conclusion of Theorem 3.4.8 at every point of the **complement** of $\mathcal{C}_\alpha$. Show that for any point $x \in \mathcal{C}_\alpha$, there exists a sequence of intervals I_n containing x such that $x \in I_n$, $|I_n| \to 0$ as $n \to \infty$, and $|I_n \cap \mathcal{C}_\alpha|/|I_n| \to 1$

as $n \to \infty$.

Solution. Fix α. First of all, $\mathcal{C}_\alpha$ is compact (it is contained in $[0,1]$ and is defined to be a countable intersection of sets which are clearly closed). Thus its complement is open. If $x \notin \mathcal{C}_\alpha$ then there exists $\delta > 0$ such that the interval $I_\delta = (x-\delta, x+\delta)$ is likewise contained in the complement of $\mathcal{C}_\alpha$. Thus $\frac{|I_\delta \cap \mathcal{C}_\alpha|}{|I_\delta|} = 1$. Moreover the same holds for any subinterval of I_δ. Thus the conclusion of the theorem holds for *every* $x \notin \mathcal{C}_\alpha$.

Regard $\mathcal{C}_\alpha$ as $\cap_{k=0}^\infty C_k$ where C_k is a union of 2^k pairwise disjoint closed intervals J_i^k of a certain common length ℓ_k. Each of the 2^k intervals of which C_k is composed is the union of 2 intervals from C_{k+1} together with an open interval of length $\alpha 3^{-k-1}$.

If $x \in \mathcal{C}_\alpha$ then for each k there exists a unique index $i = i(x)$ such that $x \in J_{i(x)}^k$. Fix x and set $I_k = J_{i(x)}^k$. The lengths of these intervals tend to zero, and each of them contains x. I claim that $\frac{|\mathcal{C}_\alpha \cap I_k|}{|I_k|} \to 1$ as $k \to \infty$.

Note first that $|J_i^k| = 2^{-k} \cdot (1-\alpha)$ for all k, i. This holds by symmetry: For any two indices i, j, $\mathcal{C}_\alpha \cap J_i^k$ is a translate of $\mathcal{C}_\alpha \cap J_j^k$, so these two portions of $\mathcal{C}_\alpha$ have identical Lebesgue measures. Since $\mathcal{C}_\alpha$ is the disjoint union of 2^k such portions, $J_i^k = 2^{-k}|\mathcal{C}_\alpha| = (1-\alpha)2^{-k}$.

Now we need to calculate $|I_k|$, that is, the length of any J_i^k. This equals $2^{-k}\sum_j |J_j^k|$. Now $\sum_j |J_j^k| = |C_k|$ is one minus the sum of all the intervals which were deleted from $[0,1]$ to form C_k:

$$\sum_j |J_j^k| = 1 - \alpha/3 - 2\alpha/3^2 - 4\alpha/3^3 - \cdots - 2^{k-1}\alpha/3^k = 1 - \frac{\alpha}{3}\sum_{n=0}^{k-1}(2/3)^n$$
$$= 1 - \frac{\alpha}{3}\frac{1-(2/3)^k}{1-(2/3)} = 1 - \alpha(1-(2/3)^k) = 1 - \alpha + (2/3)^k\alpha.$$

Thus
$$\frac{|\mathcal{C}_\alpha \cap I_k|}{|I_k|} = \frac{2^{-k}(1-\alpha)}{2^{-k}(1-\alpha+(2/3)^k\alpha)} = \frac{1-\alpha}{1-\alpha+(2/3)^k\alpha}.$$

As $k \to \infty$, this converges rapidly to 1. □

IX.D In any measure space $(E, \mathcal{A}, \mu)$ satisfying $\mu(E) < \infty$ let $\{f_n\}$ be a sequence of measurable functions, which converge almost everywhere to a limit f. Suppose moreover that f is finite μ-a.e.[1] Egoroff's theorem says that for any $\varepsilon > 0$, there exists a measurable set A satisfying $\mu(E \setminus A) < \varepsilon$ such that $f_n \to f$ *uniformly* on A. Prove this.

Solution. Consider any $k \in \mathbb{N}$. Define

$$C_m = \{x : \exists n \geq m \text{ such that } |f_n(x) - f(x)| > 2^{-k}\} = \cup_{n \geq m}\{x : |f_n(x) - f(x)| > 2^{-k}\}.$$

[1] There was an error in the original problem statement; without this extra hypothesis, and with the usual definition of uniform convergence, the conclusion may not hold. Consider for instance the example $f_n \equiv n$ on $E = [0,1]$. No matter how large n is, one never has $|f_n(x) - f(x)| < \varepsilon$.

These are countable unions of measurable sets (f is measurable since it is an almost-everywhere limit of a sequence of measurable functions), so are measurable. Moreover they are nested: $C_m \supset C_{m+1}$ for all m. If $x \in \cap_{m=1}^{\infty} C_m$, *and if* $f(x)$ *is finite*, then $f_n(x)$ does not converge to $f(x)$ as $n \to \infty$. Therefore $\cap_m C_m$ is a (measurable) subset of a null set, hence is itself a null set.[2] Since $\mu(C_1) \le \mu(E) < \infty$, and since the sets C_m are nested, $\mu(C_m) \to \mu(\cap_m C_m) = 0$ by a basic fact established earlier in the course. Therefore for any $\delta > 0$ there exists $N(k,\delta) < \infty$ such that $\mu(C_{N(k,\delta)}) < \delta$. By setting $\delta = 2^{-k}$, we obtain a set $B_k = C_{N(k,2^{-k})}$ satisfying $\mu(B_k) < 2^{-k}$, and an integer $N_k = N(k, 2^{-k})$, such that whenever $x \notin B_k$, $|f_n(x) - f(x)| \le 2^{-k}$ for all $n \ge N_k$.

From this last condition it is immediate that $f_n \to f$ uniformly on $E \setminus \cup_{k=1}^{\infty} B_k$. On the other hand $\mu(\cup_k B_k) \le \sum_k \mu(B_k) = \sum_{k=1}^{\infty} 2^{-k} = 1$.

This isn't sufficient; we wanted uniform convergence on the complement of a set of arbitrarily small measure. But it is equally immediate from the construction that $f_n \to f$ uniformly on $E \setminus \cup_{k=M}^{\infty} B_k$, for any positive integer M. Now $\mu(\cup_{k \ge M} B_k) \le 2^{1-M}$, and this can be made arbitrarily small by choosing M arbitrarily large. □

IX.E Use Egoroff's theorem to give an alternative proof of the bounded convergence theorem: If $\mu(E) < \infty$, if $f_n \to f$ μ-a.e. on E, if f_n are all measurable, and if there exists $M < \infty$ such that $|f_n(x)| \le M$ for all n and all $x \in E$, then $\int_E f_n \, d\mu \to \int_E f \, d\mu$.

Solution. This is very easy. First of all if f_n, f are measurable functions on a set A, if $f_n \to f$ uniformly on A, and if $\mu(A) < \infty$, then $|\int_A f_n \, d\mu - \int_A f \, d\mu| \le \int_A |f_n - f| \, d\mu$. Given any $\varepsilon > 0$, choose N so that $|f_n(x) - f(x)| < \varepsilon$ for all $n \ge N$. Then for all such n, $|\int_A f_n \, d\mu - \int_A f \, d\mu| \le \int_A \varepsilon \, d\mu = \varepsilon \mu(A)$. Since $\mu(A)$ is finite, this establishes convergence of $\int_A f_n \, d\mu$ to $\int_A f \, d\mu$.

Now if $|f_n(x)| \le M < \infty$ for all n, then the same holds for f almost everywhere. Thus the hypotheses of Egoroff's theorem are satisf9ied. Let $\varepsilon > 0$ be arbitrary. According to Egoroff, there exists a measurable set A such that $f_n \to f$ uniformly on A, and $\mu(A) < \varepsilon$. Then

$$\begin{aligned}
\limsup_{n\to\infty} \Big| \int_E f_n - \int_E f \Big| &\le \limsup \Big| \int_A f_n - \int_A f \Big| + \limsup \Big| \int_{E \setminus A} f_n - \int_{E \setminus A} f \Big| \\
&\le 0 + \limsup \int_{E \setminus A} |f_n| + \limsup \int_{E \setminus A} |f| \\
&\le 2M \mu(E \setminus A) \\
&< 2M\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary and M is finite and independent of ε, this concludes the proof. □

I plan to provide solutions to parts of the two problems from the text later this week.

[2] "Null set" is a common synonym for a set of measure zero.