Mathematics 105 — Spring 2004 — M. Christ
Problem Set 9 — Solutions to Selecta, part 2

3.3.21(i) Let $\mathcal{K}$ be a family of measurable functions on a measure space $(E, \mathcal{B}, \mu)$. Show that $\mathcal{K}$ is uniformly μ-integrable if it is uniformly μ-absolutely continuous and satisfies $\sup_{f\in\mathcal{K}} \|f\|_{L^1(\mu)} < \infty$. Conversely, show that if $\mathcal{K}$ is uniformly μ-integrable then it is uniformly μ-absolutely continuous. Show that if in addition $\mu(E) < \infty$, then $\sup_{f\in\mathcal{K}} \|f\|_{L^1(\mu)} < \infty$.
Solution. Suppose that $\mathcal{K}$ is uniformly μ-absolutely continouus and satisfies $\sup_{f\in\mathcal{K}} \|f\|_{L^1(\mu)} = C < \infty$. For any positive number $R < \infty$ and any $f \in \mathcal{K}$, $\mu(\{x : |f(x)| \ge R\}) \le R^{-1}\|f\|_{L^1} \le CR^{-1}$ by Markov's inequality. Let $\varepsilon > 0$. By hypothesis there exists $\delta > 0$ such that $\int_A |f|\,d\mu < \varepsilon$ for all $A \in \mathcal{B}$ satisfying $\mu(A) < \delta$. If R is chosen to be sufficiently large that $C/R < \delta$ then the set $A = \{x : |f(x)| \ge R\}$ satisfies $\mu(A) < \delta$ and hence $\int_{\{x:|f(x)|\ge R\}} |f|\,d\mu < \varepsilon$. □

Conversely suppose that $\mathcal{K}$ is uniformly μ-integrable. For any measurable set A with finite measure,

$$\int_A |f|\,d\mu \le \int_{A\cap\{|f|>R\}} |f|\,d\mu + \int_{A\cap\{|f|\le R\}} |f|\,d\mu \le \int_{\{|f|>R\}} |f|\,d\mu + R\mu(A) \qquad (1)$$

Given $\varepsilon > 0$, choose $R < \infty$ so that for all $f \in \mathcal{K}$, $\int_{\{x\in E:|f(x)|>R\}} |f|\,d\mu < \frac{1}{2}\varepsilon$; such an R exists because $\mathcal{K}$ is uniformly μ-integrable. Then choose $\delta > 0$ such that $R\delta < \frac{1}{2}\varepsilon$. Plugging this information into (1) we find that whenever $\mu(A) < \delta$, $\int_A |f|\,d\mu < \varepsilon$ for all $f \in \mathcal{K}$. Thus $\mathcal{K}$ is μ-absolutely continuous. □

Finally suppose that $\mu(E) < \infty$ and that $\mathcal{K}$ is uniformly μ-integrable. Choose R such that for any $f \in \mathcal{K}$, $\int_{\{x:|f(x)|\ge R\}} |f|\,d\mu \le 1$. By (1), applied to $A = E$, we get $\int_E |f|\,d\mu \le 1 + R\mu(E)$ for any $f \in \mathcal{K}$. The right-hand side $1 + R\mu(E)$ is a finite constant. □

3.3.21(ii) Suppose that for some $\delta > 0$, $\sup_{f\in\mathcal{K}} \int |f|^{1+\delta}\,d\mu < \infty$. Show that $\mathcal{K}$ is then uniformly μ-integrable.
Solution. Define $C = \sup_{f\in\mathcal{K}} \int |f|^{1+\delta}\,d\mu < \infty$. For any $f \in \mathcal{K}$ and any $R < \infty$, let $A = \{x : |f(x)| \ge R\}$. Now $|f(x)| \le R^{-\delta}|f(x)|^{1+\delta}$ for all $x \in A$. Therefore

$$\int_A |f|\,d\mu \le \int_A R^{-\delta}|f|^{1+\delta}\,d\mu \le R^{-\delta}\int_E |f|^{1+\delta}\,d\mu \le CR^{-\delta} \underset{R\to\infty}{\longrightarrow} 0. \qquad \square$$

3.3.21(iii) Let $\{f_n\}_{n=1}^\infty \subset L^1(\mu)$ be given. Show that if $\mu(E) < \infty$, if $f_n \to f$ in measure, and if $\{f_n\}$ is uniformly $\mu - \mathit{integrable}$, then $f_n \to f$ in $L^1(\mu)$.
Solution. Suppose that $\mu(E) < \infty$, $f_n \to f$ in measure, and $\{f_n\}$ is uniformly μ-integrable. By part (i), $\sup_n \|f_n\|_{L^1} < \infty$, We know that some subsequence $\{f_{n_k}\}$ converges to f μ-almost everywhere, and then it follows by Fatou's lemma that

$$\int |f|\,d\mu = \int \liminf_{k\to\infty} |f_{n_k}|\,d\mu \le \liminf_{k\to\infty} |f_{n_k}|\,d\mu \le \sup_n \|f_n\|_{L^1} < \infty,$$

so $f \in L^1$ as well. Therefore f satisfies the conclusion of problem 3.3.16, so $\{f_n\} \cup \{f\}$ is uniformly μ-absolutely continuous by part (i) of this problem.

Let $\varepsilon > 0$. Choose δ so $\int_A |f_n| \, d\mu < \varepsilon$ for all n and $\int_A |f| \, d\mu < \varepsilon$, for all $A \in \mathcal{B}$ satisfying $\mu(A) < \delta$. Choose N so that for all $n \geq N$, $\mu(\{x : |f_n(x) - f(x)| > \varepsilon\} < \delta$.

Consider any $n \geq N$, and let $A = \{x : |f_n(x) - f(x)| \leq \varepsilon\}$, which is of course a measurable set.

$$\begin{aligned}\|f_n - f\|_{L^1} &= \int_A |f_n - f| \, d\mu + \int_{E \setminus A} |f_n - f| \, d\mu \\ &\leq \varepsilon\mu(A) + \int_{E \setminus A} |f_n| \, d\mu + \int_{E \setminus A} |f| \, d\mu.\end{aligned}$$

Since $\mu(E \setminus A) < \delta$ by our choice of N, $\int_{E \setminus A} |f_n| \, d\mu < \varepsilon$ and likewise $\int_{E \setminus A} |f| \, d\mu < \varepsilon$. Thus $\|f_n - f\|_{L^1} < 2\varepsilon + \mu(A)\varepsilon \leq 2\varepsilon + \mu(E)\varepsilon$. Since $\mu(E)$ is finite and $\varepsilon > 0$, this concludes the proof. $\square$

3.3.22 Let $(E, \mathcal{B}, \mu)$ be a finite measure space. Show that $f_n \to f$ in μ-measure if and only if $\int \min(|f_n - f|, 1) \, d\mu \to 0$.

Solution. First recall that the minimum of any two measurable functions is again a measurable function, so the integrals in question are well-defined. Suppose first that $\int \min(|f_n - f|, 1) \, d\mu \to 0$. To show that $f_n \to f$ in measure it suffices to show that for any $\varepsilon \in (0, 1)$, $\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \to 0$ as $n \to \infty$; this implies the same conclusion for all larger ε. But now $|f_n(x) - f(x)| > \varepsilon$ if and only if $\min(|f_n(x) - f(x)|, 1) < \varepsilon$. Therefore by Markov's inequality

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \leq \varepsilon^{-1} \| \min(|f_n - f|, 1)\|_{L^1}.$$

As $n \to \infty$, this tends to zero. (The hypothesis that $\mu(E) < \infty$ is not needed for this implication.)

Now suppose conversely that $f_n \to f$ in measure. Let $\varepsilon > 0$ be given. Choose N so that whenever $n \geq N$, the set $A_n = \{x : |f_n(x) - f(x)| \leq \varepsilon\}$ satisfies $\mu(E \setminus A_n) < \varepsilon$. Now for any $n \geq N$,

$$\begin{aligned}\int \min(|f_n - f|, 1) \, d\mu &= \int_{A_n} \min(|f_n - f|, 1) \, d\mu + \int_{E \setminus A_n} \min(|f_n - f|, 1) \, d\mu \\ &\leq \varepsilon\mu(A_n) + \int_{E \setminus A_n} 1 \, d\mu \leq \varepsilon\mu(A_n) + \mu(E \setminus A_n) \leq \varepsilon\mu(E) + \varepsilon.\end{aligned}$$

$\square$