Mathematics 105 — Spring 2004 — M. Christ

Problem Set 8 Solutions (selecta)

VIII.A Let $f : [a,b] \times [c,d] \to \mathbb{R}$ and suppose that for each $y \in [c,d]$, $[a,b] \ni x \mapsto f(x,y)$ is an integrable[1] function with respect to Lebesgue measure. Suppose also that the partial derivative $\partial f/\partial y$ exists for all $(x,y) \in [a,b] \times (c,d)$, and that its absolute value is bounded by a certain finite constant, which is independent of x,y. Prove that for each $y \in (c,d)$, $x \mapsto \partial f(x,y)/\partial y$ is a measurable function of $x \in [a,b]$. Prove that for all $y \in (c,d)$,

$$\frac{d}{dy}\int_{[a,b]} f(x,y)\,d\lambda(x) = \int_{[a,b]} \frac{\partial f(x,y)}{\partial y}\,d\lambda(x).$$

Solution. Fix $y \in (c,d)$. Then $\partial f(x,y)/\partial y = \lim_{n\to\infty} g_n(x)$ where $g_n(x) = \frac{f(x,y+n^{-1})-f(x,y)}{1/n}$. Both $x \mapsto f(x,y)$ and $x \mapsto f(x,y+n^{-1})$ are measurable functions of $x \in [a,b]$, by hypothesis. Since linear combinations of measurable functions are measurable, so is each g_n. We are given that $g_n(x) \to \partial f(x,y)/\partial y$ for every $x \in [a,b]$. Since a limit of measurable functions is measurable, it follows that $\partial f(x,y)/\partial y$ is a measurable function of x.

For the second conlusion let M be a finite constant such that $|\partial f(x,y)/\partial y| \le M$ for all $(x,y) \in [a,b] \times (c,d)$. Consider any sequence (h_n) of nonzero real numbers tending to zero. Redefine $g_n(x,y) = \frac{f(x,y+h_n)-f(x,y)}{h_n}$. By the mean value theorem, for each $(x,y) \in [a,b] \times (c,d)$ there exists $c = c(x,y,h_n)$ such that $g_n(x,y) = (\partial f/\partial y)(x,c)$. Thus $|g_n(x,y)| \le M$ for all $(x,y) \in [a,b] \times (c,d)$. Moreover $g_n(x,y) \to \partial f(x,y)/\partial y$ as $n \to \infty$, for all $(x,y) \in [a,b] \times (c,d)$, by definition of the partial derivative.

Fix $y \in (c,d)$. We may apply the dominated convergence theorem to the functions $x \mapsto g_n(x,y)$, using the integrable function $M\chi_{[a,b]}$ as the Lebesgue dominator, to conclude that $\int_{[a,b]} g_n(x,y)\,d\lambda(x) \to \int_{[a,b]} \frac{\partial f(x,y)}{\partial y}\,d\lambda(x)$.

Define now $F(y) = \int_{[a,b]} \frac{\partial f(x,y)}{\partial y}\,d\lambda(x)$. The associated difference quotients are

$$\frac{F(y+h)-F(y)}{h} = h^{-1}\Big(\int_{[a,b]} f(x,y+h)\,d\lambda(x) - \int_{[a,b]} f(x,y)\,d\lambda(x)\Big) = \int_{[a,b]} \Big(\frac{f(x,y+h)-f(x,y)}{h}\Big) d\lambda(x).$$

Thus we've proved that

$$\lim_{n\to\infty} \frac{F(y+h_n)-F(y)}{h_n} = \int_{[a,b]} \frac{\partial f(x,y)}{\partial y}\,d\lambda(x).$$

Since this holds for any sequence (h_n) of nonzero real numbers tending to zero, with a limit independent of the sequence, we've proved that $\lim_{h\to 0} \frac{F(y+h)-F(y)}{h}$ exists and equals $\int_{[a,b]} \frac{\partial f(x,y)}{\partial y}\,d\lambda(x)$. □

VIII.B Let $(E,\mathcal{A},\mu)$ be any measure space and let $f \in L^1(E,\mu)$. Consider the mapping $L_f(A) = \int_A f\,d\mu$, where the domain is the collection of all sets $A \in \mathcal{A}$. Give an example to show that, for general measure spaces, the parameter δ in problem 3.3.16 cannot be chosen to depend on ε and $\|f\|_{L^1}$ alone.

Solution. To prove that δ cannot be chosen to depend on ε and $\|f\|_{L^1}$ alone, for a particular measure space $(E,\mathcal{A},\mu)$, it suffices to show that for any $c > 0$ there exists $\varepsilon > 0$ such that for any sufficiently small $\delta > 0$, there exist a nonnegative function $f \in L^1$ satisfying $\|f\|_{L^1} = c$ and a set $A \in \mathcal{A}$ satisfying $\mu(A) < \delta$ such that $\int_A f\,d\mu > \varepsilon$. I'll prove this under one assumption on $(E,\mathcal{A},\mu)$: For any $\delta > 0$ there exists $A \in \mathcal{A}$ satisfying $0 < \mu(A) < \delta$. (For measure spaces which don't satisfy

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[1] I carelessly wrote "measurable" in the original problem statement. We need to assume that the integrals that appear in the problem statement are defined!

this assumption, there is no such counterexample; if $\delta > 0$ is sufficiently small then $\mu(A) < \delta$ implies $\mu(A) = 0$ and hence $\int_A |f|\,d\mu = 0 < \varepsilon$ for any $f \in L^1$ and any $\varepsilon > 0$.)

Let $c > 0$ be given. Define $\varepsilon = c/2$. Let $\delta > 0$ be given. Choose any set $A \in \mathcal{A}$ satisfying $0 < \mu(A) < \delta$, and define $f = c\mu(A)^{-1}\chi_A$. Then f is nonnegative, $\|f\|_{L^1} = c$, yet $\int_A f = c\mu(A)^{-1}\mu(A) = c > \varepsilon$. □

3.3.16 Let $(E, \mathcal{A}, \nu)$ be a measure space $f \in L^1$ be a nonnegative integrable function. Define $\mu(A) = \int_A f\,d\nu$ for all $A \in \mathcal{A}$. (i) Show that μ is a measure. (ii) Prove that μ is *absolutely continuous*, that is, that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_A f\,d\nu < \varepsilon$ for all measurable sets satisfying $\nu(A) < \delta$.

Solution. (i) $\mu(\emptyset) = \int_\emptyset f\,d\nu = 0$ since $\nu(\emptyset) = 0$, $\mu(A)$ is defined for all elements A of the σ-algebra $\mathcal{A}$, and $\mu(A) \ge 0$ for all $A \in \mathcal{A}$. So all that remains to be proved is that if sets $A_j \in \mathcal{A}$ are pairwise disjoint, then $\mu(\cup_j A_j) = \sum_j \mu(A_j)$. (Here j ranges either over $\mathbb{N} = \{1, 2, 3, \cdots\}$ or over $\{1, 2, \cdots, N\}$ for some $N \in \mathbb{N}$. For the sake of simplicity of notation, I'll discuss only the former case.)

Define $g_j = f\chi_{A_j}$. For any positive integer n, the characteristic function of $\cup_{j=1}^N A_j$ equals $\sum_{j=1}^N \chi_{A_j}$. Therefore

$$\mu(\cup_{j=1}^n A_j) = \int_{A_1\cup\cdots\cup A_n} f\,d\nu = \int f\cdot\chi_{A_1\cup\cdots\cup A_n}\,d\nu = \int f\cdot\sum_{j=1}^n \chi_{A_j}\,d\nu = \sum_{j=1}^n \int_{A_j} f\,d\nu = \sum_{j=1}^n \mu(A_j)$$

because the integral of any finite sum of integrable functions equals the sum of the corresponding integrals.

To prove the corresponding identity for infinite sums, let $A = \cup_{j=1}^\infty A_j$ and note that

$$\sum_{j=1}^n \chi_{A_j}(x) = \chi_{A_1\cup\cdots\cup A_n}(x) \to \chi_A(x) \text{ as } n \to \infty \text{ for all } x \in E.$$

Define $g_j = f\cdot\chi_{A_1\cup\cdots\cup A_n}$. These are measurable functions, $0 \le g_n(x) \le g_{n+1}(x) \le f\chi_A(x)$ for all x, and we have just shown that $\lim_{n\to\infty} g_n(x) = f\chi_A(x)$ for all x. Therefore by the monotone convergence theorem, $\int g_n\,d\nu \to \int f\chi_A\,d\nu$ as $n \to \infty$.

The right-hand side here is $\mu(A)$, while the left-hand side is $\mu(\cup_{j=1}^n A_j) = \sum_{j=1}^n \mu(A_j)$. Thus we have shown that $\lim_{n\to\infty} \sum_{j=1}^n \mu(A_j) = \mu(A)$, that is, that $\sum_{j=1}^\infty \mu(A_j) = \mu(\cup_{j=1}^\infty A_j)$. □

(ii): To prove that $\mu(A) \to 0$ as $\nu(A) \to 0$ (in the precise specified in the problem statement) consider first the case where f is bounded. Fix some finite constant such that $0 \le f(x) \le M$ for all $x \in E$. Then $\int_A f\,d\nu \le \int_A M\,d\nu = M\nu(A)$, so the conclusion obviously holds.

Next let f be an arbitrary nonnegative integrable function, and let $\varepsilon > 0$ be given. I'll show in the next paragraph that there exists a bounded nonnegative measurable function g such that $\|f - g\|_{L^1} < \varepsilon/2$. Granting this, choose $M \in (0, \infty)$ so that $0 \le g(x) \le M$ for all x. (M depends on g, but we have already chosen g in terms of ε.) Define $\delta = \varepsilon/2M$. If $A \in \mathcal{A}$ satisfies $\nu(A) < \delta$ then

$$\mu(A) = \int_A f\,d\nu = \int_A g\,d\nu + \int_A (f - g)\,d\nu.$$

The two terms on the right-hand side are both favorable, but for different reasons. Firstly

$$0 \le \int_A g\,d\nu \le M\nu(A) \le M\delta = \varepsilon/2.$$

Secondly

$$|\int_A (f - g)\,d\nu| \le \int_A |f - g|\,d\nu \le \int_E |f - g|\,d\nu = \|f - g\|_{L^1} < \varepsilon/2.$$

All together we thus have $0 \le \mu(A) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$, as desired.

It remains to prove the claim. For each $n \in \mathbb{N}$ define $g_n(x) = f(x)$ whenever $f(x) \le n$, and $= 0$ whenever $f(x) > n$. Then each g_n is measurable, $0 \le g_n(x) \le g_{n+1}(x) \le f(x)$ for all $x \in E$, and $g_n(x) \to f(x)$ for all x such that $f(x)$ is finite (but not when $f(x) = +\infty$). Since $f \in L^1$, we know that $f(x) < \infty$ for ν-a.e. $x \in E$. Thus $g_n \to f$ ν-a.e. Therefore by the monotone convergence theorem, $\int g_n \, d\nu \to \int f \, d\mu$. Since $f \in L^1$ and $0 \le g_n \le f$, each g_n belongs to L^1. Since $g_n \le f$, we have $\|f - g_n\|_{L^1} = \int (f - g_n) \, d\nu = \int f \, d\nu - \int g_n \, d\nu$, and we have just shown that this difference approaches 0 as $n \to \infty$. Thus $\|f - g_n\|_{L^1} \to 0$ as $n \to \infty$. Since each function g_n is bounded, this proves the claim. □

3.3.17 Let $0 \le f \in L^1$ for some measure space $(E, \mathcal{A}, \mu)$. Show that $\lambda \cdot \mu(\{x : f(x) \ge \lambda\}) \to 0$ as $\lambda \to \infty$.

Solution. Note that Markov's inequality seems to be related, but certainly doesn't give this conclusion. Define $f_t(x) = f(x)$ if $f(x) \le t$ and $= 0$ otherwise. Then f_t is measurable, and $\|f - f_t\|_{L^1} \to 0$ as $t \to +\infty$; this was essentially proved in the last paragraph of the preceding solution. (There we considered only integers t but since $(f - f_t) \ge (f - f_s) \ge 0$ whenever $0 \le t \le s \in \mathbb{R}$, the result for the limit as $t \to \infty$ through $\mathbb{R}$ follows from the result for integers t by a sandwiching argument.)

Note that $f(x) > \lambda$ if and only if $(f - f_\lambda)(x) > \lambda$; $f - f_\lambda$ is f where $f > \lambda$, and is zero otherwise. Now invoke Markov's inequality:

$$\lambda\mu(\{x : f(x) > \lambda\}) = \lambda\mu(\{x : (f - f_\lambda)(x) > \lambda\}) \le \|f - f_\lambda\|_{L^1} \to 0. \qquad \square$$

3.3.17 Let $f \ge 0$ be a nonnegative measurable function on a finite measure space $(E, \mathcal{A}, \mu)$ (that is, $\mu(E) < \infty$). Show that f is integrable if and only if $\sum_{n=1}^{\infty} \mu(\{x : f(x) > n\})$ is finite.

Solution. Define $g_n(x) = 1$ if $f(x) > n$ and $= 0$ otherwise. Then g_n is measurable and nonnegative. Moreover $\sum_{n=1}^{\infty} g_n(x) \le f(x)$ for all x. Indeed, if $f(x) \le 1$ then $g_n(x) = 0$ for all n. If $f(x) \in (m, m+1]$ for some $m \in \mathbb{N}$ then $g_n(x) = 1$ for all $1 \le n \le m$, and $g_n(x) = 0$ otherwise.

Therefore $\int f \, d\mu \ge \int \sum_n g_n \, d\mu = \sum_n \int g_n \, d\mu$ (see next paragraph for justification). But $\int g_n \, d\mu = \mu(\{x : f(x) > n\})$, so if $f \in L^1$ then the infinite series in the problem statement converges.

To justify interchanging the integral and infinite sum in the preceding paragraph note that for any *nonnegative* measurable functions h_n, the functions $H_n = \sum_{k=1}^{n} h_n$ satisfy the hypotheses of the monotone convergence theorem, so

$$\int \sum_{k=1}^{\infty} h_k \, d\mu = \int \lim_{n\to\infty} H_n \, d\mu = \lim_{n\to\infty} \int H_n \, d\mu = \lim_{n\to\infty} \int \sum_{k=1}^{n} h_k \, d\mu = \lim_{n\to\infty} \sum_{k=1}^{n} \int h_k \, d\mu = \sum_{k=1}^{\infty} \int h_k \, d\mu;$$

the last equality holds because each term $\int h_k \, d\mu$ is nonnegative, so the series $\sum_{k=1}^{\infty} \int h_k \, d\mu$ either converges to some finite sum, or diverges to $+\infty$; either way, the infinite series is defined as an element of $[0, +\infty]$. Note that we did not need to assume that $\mu(E)$ is finite.

It remains to prove the converse; here we do assume $\mu(E) < \infty$. The above discussion shows that for any $x \in E$, $f(x) \le 1 + \sum_{n=1}^{\infty} g_n(x)$. It shows also that $\sum_{n=1}^{\infty} \mu(\{x : f(x) > n\}) = \sum_{n=1}^{\infty} \int g_n \, d\mu = \int (\sum_{n=1}^{\infty} g_n) \, d\mu$. Now

$$\int f \, d\mu \le \int \big(1 + \sum_n g_n\big) \, d\mu = \int 1 \, d\mu + \sum_n \int g_n \, d\mu \le \mu(E) + \sum_n \mu(\{x : f(x) > n\}.$$

Since $\mu(E) < \infty$, convergence of the infinite series on the right thus implies finiteness of $\int f \, d\mu$. (And we are assuming that $f \ge 0$, so finiteness of the integral is the same thing as integrability of f.) □

3.3.19 Let $[a, b]$ be a closed bounded interval in $\mathbb{R}^1$, and let $f : [a, b] \to \mathbb{R}$ be continuous. Show that the Riemann integral $\int_a^b f(x) \, dx$ equals the Lebesgue integral $\int_{[a,b]} f \, d\lambda$, where λ denotes Lebesgue measure on $\mathbb{R}^1$.

Apology. In order both to simplify notation and focus on the most essential ideas I've decided to write the solution out only in dimension one, and hence have formulated the result that way.

Solution. Let $N \in \mathbb{N}$ be arbitrary. For $0 \le j \le N$ define $t_j = a + j(b-a)/N$. Define the intervals $I_j = [t_{j-1}, t_j)$ for $1 \le j < N$ and $I_N = [t_{N-1}, t_N]$. (Both t_j and I_j depend also on N, and this dependence could be indicated by writing t_j^N and I_j^N, but I hope there's little danger of confusion now that the issue has been pointed out.) For each N, j choose $x_j \in I_j$, and set $c_j = f(x_j)$. (Again, these depend also on N.) Since f is Riemann integrable, $\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{j=1}^N c_j(b-a)/N$.

Consider the function $f_N(x) = \sum_{j=1}^N c_j \chi_{I_j}(x)$. From the continuity of f it follows easily that $f_N(x) \to f(x)$ as $N \to \infty$, for all $x \in [a,b]$. Therefore by the bounded convergence theorem, the Lebesgue integral of f_N converges to the Lebesgue integral of f over $[a,b]$ as $N \to \infty$.

Now the Lebesgue integral of f_N equals $\sum_{j=1}^N c_j \lambda(I_j)$, which clearly equals $\sum_{j=1}^N c_j(b-a)/N$; and we've already noted that these quantities converge to the Riemann integral of f. ☐

3.3.20(i) Let f_n, g_n be measurable functions on a measure space $(E, \mathcal{A}, \mu)$. Suppose that $g_n \in L^1$, $g \in L^1$, and $g_n \to g$ in L^1 norm. Suppose that $f_n \le g_n$ μ-a.e. for all n. Prove that $\limsup_{n\to\infty} \int f_n\,d\mu \le \int \limsup_{n\to\infty} f_n\,d\mu$.

Solution. Two things are apparent: Fatou's lemma ought to come into play, and the hypothesis $f_n \le g_n$ (almost everywhere with respect to μ) can be written as $g_n - f_n \ge 0$ almost everywhere. Thus we get

$$\int \liminf (g_n - f_n)\,d\mu \le \liminf \int (g_n - f_n)\,d\mu,$$

where $\liminf$ means of course $\liminf_{n\to\infty}$. Since $\int g_n\,d\mu \to \int g\,d\mu$, the right-hand side equals $\int g\,d\mu - \limsup \int f_n\,d\mu$, which is certainly encouraging. But the left-hand side is a problem; convergence of $\{g_n\}$ to g in L^1 norm does not imply any sort of pointwise or almost-everywhere convergence, as we discussed in class.

The next thought that occurs to us is that some subsequence of $\{g_n\}$ does converge to g, almost everywhere with respect to μ. However, after passing to such a subsequence, we would be considering only some f_n, not all f_n, so we couldn't possibly reach any conclusion about $\limsup \int f_n\,d\mu$.

The fix is surprisingly easy: At the outset choose a subequence f_{n_k} such that $\lim_{k\to\infty} \int f_{n_k}\,d\mu$ equals $\limsup_{n\to\infty} \int f_n\,d\mu$. (Here we allow the possibility that the $\limsup$ equals $+\infty$, in which case we mean by this identity that $\int f_{n_k}\,d\mu \to +\infty$ as $k \to \infty$. The $\limsup$ of any sequence of extended real numbers equals the limit of some subsequence.) Next, choose a further subsequence (n_{k_j}) of (n_k) so that $g_{n_{k_j}} \to g$ μ-a.e. To simplify notation set $F_j = f_{n_{k_j}}$ and $G_j = g_{n_{k_j}}$. Then $\int F_j\,d\mu \to \limsup_{n\to\infty} \int f_n\,d\mu$, and $G_j \to g$ almost everywhere.

By invoking Fatou as above we find that

$$\int \liminf_{j\to\infty} (G_j - F_j)\,d\mu \le \liminf_{j\to\infty} \int (G_j - F_j)\,d\mu. \tag{1}$$

For any x for which $G_j(x) \to g(x)$ we have $\liminf (G_j - F_j)(x) = g(x) - \limsup F_j(x)$. On the other hand $\int G_j\,d\mu \to \int g\,d\mu$ since $G_j \to g$ in L^1 norm, so

$$\liminf \int (G_j - F_j)\,d\mu = \int g\,d\mu - \limsup \int F_j\,d\mu.$$

Insert all this information into (1). Since $g \in L^1$ we may subtract its integral from both sides and rearrange to deduce

$$\limsup \int F_j\,d\mu \le \int \limsup F_j\,d\mu. \tag{2}$$

We have arranged that the left-hand side equals $\limsup_{n\to\infty} \int f_n\,d\mu$. For the right-hand side we've made no such arrangement[2] but luck or providence is on our side:

$$\limsup_{j\to\infty} F_j(x) = \limsup_{j\to\infty} f_{n_{k_j}}(x) \le \limsup_{n\to\infty} f_n(x)$$

simply because the lim sup of a subsequence never exceeds the lim sup of the whole sequence. Inserting these two pieces of information into (2) completes the proof. $\square$

3.3.20(ii) Let the functions g_n, g satisfy the hypotheses of part (i) of this problem. Suppose that f_n is measurable, $|f_n| \le g_n$ μ-almost everywhere and that $f_n \to f$ μ-almost everywhere. Show that $\|f_n - f\|_{L^1} \to 0$.

Solution. f is measurable. If $f_n \to f$ almost everywhere, we have proved this. If $f_n \to f$ in measure then some subsequence f_{n_k} converges to f almost everywhere, whence the same conclusion.

$|f| \le g$ almost everywhere, and consequenty $f \in L^1$. Indeed, since $g_n \to g$ in L^1 norm, there exists a subsequence g_{n_k} which converges almost everywhere to g. By passing to a further sub-sub-subsequence, still denoted (n_k), we may also arrange that $f_{n_k} \to f$ almost everywhere, whence $|f_{n_k}(x)| \to |f(x)|$ for almost every x. Whenever $x \in E$ is a point for which both these convergence statements hold, and for which $|f_n(x)| \le g_n(x)$ for all n, it follows that $|f(x)| \le g(x)$. Since a countable union of null sets is again a null set, it follows that $|f(x)| \le g(x)$ for almost every x.

Now we set things up for an invocation of part (i). Define $F_n = |f_n - f|$ and $G_n = g_n + g$. Then $G_n \to G = 2g$ in L^1 norm, and $F_n \le G_n$ almost everywhere. By part (i),

$$\limsup \int |f_n - f|\, d\mu \le \int \limsup |f_n - f|\, d\mu.$$

If $f_n \to f$ almost everywhere then the right-hand side is 0, whence $\limsup \int |f_n - f|\, d\mu = 0$; this means that $\|f_n - f\|_{L^1} \to 0$.

If we merely know that $f_n \to f$ in measure then we revert to the very beginning of the argument, and choose a subsequence (f_{n_k}) such that $\|f_{n_k} - f\|_{L^1} \to \limsup_{n\to\infty} \|f_n - f\|_{L^1}$ as $k \to \infty$. Next choose a sub-subsequence, again denoted $(f_{n_{k_j}})$, which converges to f almost everywhere. Apply the result proved in the preceding paragraphs to conclude that $\limsup_{j\to\infty} \|f_{n_{k_j}} - f\|_{L^1} = 0$. But we've arranged that $\lim_{k\to\infty} \|f_{n_k} - f\|_{L^1}$ exists. Therefore it equals the corresponding lim sup over any subsequence. Thus $\limsup_{n\to\infty} \|f_n - f\|_{L^1} = \lim_{k\to\infty} \|f_{n_k} - f\|_{L^1} = 0$. $\square$

3.3.23 Let (E, ρ) be a metric space, let μ, ν be two finite measures defined on the Borel σ-algebra $\mathcal{B}_E$. Suppose that $\int \varphi\, d\mu = \int \varphi\, d\nu$ for all bounded, uniformly continuous functions $\varphi : E \to \mathbb{R}$. Show that $\mu = \nu$ (that is, $\mu(A) = \nu(A)$ for all $A \in \mathcal{B}_E$).

Solution. The natural way to attack the problem is this: Consider any $A \in \mathcal{B}_E$, and let $\varepsilon > 0$; we aim to prove that $|\mu(A) - \nu(A)| < \varepsilon$.

Note that $\chi_A \in L^1(\mu) \cap L^1(\mu)$ since both μ, ν are finite measures. We know that there exist a bounded uniformly continuous function φ such that $\int |\chi_A - \varphi|\, d\nu < \varepsilon/2$, and likewise a bounded uniformly continuous ψ such that $\int |\chi_A - \psi|\, d\mu < \varepsilon/2$. $\underline{\underline{\text{If}}}$ we had $\varphi \equiv \psi$ then we'd be in business:

$$\begin{aligned}\big|\mu(A) - \nu(A)\big| = \Big|\int \chi_A\, d\mu - \int \chi_A\, d\nu\Big| &= \Big|\int (\chi_A - \varphi)\, d\mu - \int (\chi_A - \varphi)\, d\nu\Big| \\ &\text{would equal } \Big|\int (\chi_A - \psi)\, d\mu - \int (\chi_A - \varphi)\, d\nu\Big| \quad \text{if } \varphi = \psi\end{aligned}$$

since $\int \varphi\, d\nu = \int \varphi\, d\mu$ by hypothesis. We could then deduce that

$$\big|\mu(A) - \nu(A)\big| \le \Big|\int (\chi_A - \psi)\, d\mu\Big| + \Big|\int (\chi_A - \varphi)\, d\nu\Big| \le \int |\chi_A - \psi|\, d\mu + \int |\chi_A - \varphi|\, d\nu < \varepsilon.$$

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[2] And we can't; if you try to do this you could well end up choosing a completely different sequence of indices for each point $x \in E$, and if E is uncountable this is hopeless ...

But why can φ, ψ be chosen to be the same? One way to approach this is to read the proof in which they were constructed, and to verify step-by-step that one and the same construction works, regardless of the underlying measure. This is indeed the case (I leave it to you to check this if you wish.). But a simple little trick saves us from having to do this: Define a third measure, λ, by setting $\lambda(A) = \mu(A) + \nu(A)$ for all $A \in \mathcal{B}_E$. It is easily verified that λ is indeed a measure. Moreover $L^1(\nu) \cap L^1(\mu) \subset L^1(\lambda)$, and for any $f \in L^1(\mu) \cap L^1(\nu)$, $\int f \, d\lambda = \int f \, d\mu + \int f \, d\nu$. These things hold for simple functions by the definitions of λ and of the integral of a simple function. They then follow for nonnegative functions by the corresponding definition, and then for general measurable functions. Details are left to the patient reader.[3]

Now $\chi_A \in L^1(\lambda)$, since $\int \chi_A \, d\lambda = \lambda(A) = \mu(A) + \nu(A) < \infty$. Therefore we may apply the approximation theorem for the measure λ to conclude that there exists a single continuous function φ such that $\int |\chi_A - \varphi| \, d\lambda < \varepsilon$. This means precisely that

$$\int |\chi_A - \varphi| \, d\mu + \int |\chi_A - \varphi| \, d\nu < \varepsilon.$$

The above reasoning then applies! $\square$

Comment. This notion of the *sum* of two measures arises often in other situations.

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[3] We have here an instance of the principle of conservation of work; the device of defining λ changed the form of the details we need to verify but didn't completely eliminate them.