**Mathematics 105, Spring 2004 — Problem Set VII Solutions**

**#3.2.17** Let $(E, \mathcal{A}, \mu)$ be a measure space and $f, g$ be measurable functions. Suppose that $f, g$ are integrable, and that $\int_A f\, d\mu = \int_A g\, d\mu$ for all $A \in \mathcal{A}$. Show that $f \overset{\mu}{\sim} g$. (This is the more difficult half of the problem.)

**Solution.** Let $h = f - g$, which is integrable since $f, g$ are. We're given that $\int_A h\, d\mu = 0$ for all $A \in \mathcal{A}$. For any $n \in \mathbb{N}$ define $A_n = \{x : h(x) \geq 1/n\}$. Then $\int_{A_n} h\, d\mu \geq \int_{A_n} n^{-1}\, d\mu = n^{-1}\mu(A_n)$. Since we're given that the integral is zero, we conclude that $\mu(A_n) = 0$ for all $n$. But $\{x : h(x) > 0\} = \cup_n A_n$ is thus a countable union of sets of measure zero, so its measure is likewise zero. The same reasoning (or replace $h$ by $-h$) applies to $\{x : h(x) < 0\}$. □

*Confession.* This solution is a bit sloppy, but in a manner which is conventional in this subject. $f(x) - g(x)$ may be undefined for a set of points having $\mu$-measure zero. By $h(x) = f(x) - g(x)$ is meant any measurable function which equals $f(x) - g(x)$ at every point where the difference is well-defined. The values of $h$ on the exceptional set where the difference is not defined have no effect on integrals of $h$, since this set has measure zero.

**VII.A** Recall the Cantor set $\mathcal{C} = \cap_{k=0}^{\infty} \mathcal{C}_k$, where $\mathcal{C}_k$ is the union of $2^k$ closed intervals $\{I_j^k : 1 \leq j \leq 2^k\}$, each of which has length $3^{-k}$, and $\mathcal{C}_{k+1}$ is the subset of $\mathcal{C}_k$ obtained by deleting the (open) middle third of each interval $I_j^k$. Thus $\mathcal{C}$ has Lebesgue measure equal to zero. Recall also that the open set $[0,1] \setminus \mathcal{C}$ is a disjoint union of infinitely many open intervals, the sum of whose lengths equals one.

Let's write $I_j^k = [a_j^k, b_j^k]$, and order them from left to right, so that $a_1^k = 0$, $b_j^k < a_{j+1}^k$ for all $j$, and $b_{2^k}^k = 1$. Then each $a_j^k$ is a certain integer multiple of $3^{-k}$, and $b_j^k = a_j^k + 3^{-k}$. The open intervals mentioned above are all the intervals $(b_j^k, a_{j+1}^k)$, where $k, j$ range over all allowed values. Their lengths vary from $3^{-k}$ to $3^{-1}$.

Define a sequence of functions $f_k : [0,1] \to [0,1]$ as follows: $f_k$ is the unique continuous piecewise-linear function which maps 0 to 0, has constant slope $(3/2)^k$ on each interval $I_j^k$, and is constant on the interval $[b_j^k, a_{j+1}^k]$ for all $1 \leq j < 2^k$.

Figure 1: The graphs of $f_1$ and $f_2$, superimposed.

(i) Prove that the sequence $\{f_k\}$ converges uniformly to some continuous function $f : [0,1] \to [0,1]$.

**Solution.** Let $x \in [0,1]$. I'll show in a moment that

$$|f_{k+1}(x) - f_k(x)| \leq 3 \cdot 2^{-k} \quad \text{for all } k. \tag{1}$$

This implies that $\{f_k(x)\}$ is a Cauchy sequence for each $x$, for whenever $k < i$,

$$\begin{aligned}|f_k(x) - f_i(x)| &\le |f_k(x) - f_{k+1}(x)| + |f_{k+1}(x) - f_{k+2}(x)| + \cdots + |f_{i-1}(x) - f_i(x)| \\ &\le \sum_{j=k}^{i-1} 3 \cdot 2^{-j} \le 6 \cdot 2^{-k}.\end{aligned}$$

Thus $\lim_{k\to\infty} f_k(x)$ exists, and we define it to be $f(x)$. From the last inequality established above it follows that $|f_k(x) - f(x)| \le 6 \cdot 2^{-k}$ for all $k$, so $f_k \to f$ uniformly on $[0, 1]$, so $f$ is continuous (a general theorem from Math 104 asserts that any uniform limit of continuous functions is continuous).

Let's prove (1). I claim that for any $n$,

$$f_{n+1}(a_j^n) = f_n(a_j^n) \text{ for each of the endpoints } a_j^n. \tag{2}$$

Granting (2), we have for any $x \in I_j^n$ $|f_n(x)-f_n(a_j^n)| \le (3/2)^n|x-a_j^n| \le (3/2)^n 3^{-n} = 2^{-n}$, since over this interval, $f_n$ has constant derivative $(3/2)^n$. Similarly $|f_{n+1}(x)-f_{n+1}(a_j^n)| \le (3/2)^{n+1}|x-a_j^n| \le \frac{3}{2}2^{-n}$, since the absolute value of the derivative of $f_{n+1}$ does not exceed $(3/2)^{n+1}$. (The mean value theorem doesn't apply directly since there are two points in the interior of $I_j^n$ where $f_{n+1}$ fails to be differentiable, but since $f_{n+1}$ is piecewise linear this is still easily justified.) Thus whenever $x \in I_j^n$,

$$|f_n(x) - f_{n+1}(x)| \le |f_n(x) - f_n(a_j^n)| + |f_{n+1}(a_j^n) - f_{n+1}(x)| \le 2\tfrac{3}{2}2^{-n} = 3 \cdot 2^{-n}.$$

Now we prove (2) by induction on $j$ for each $n$. For $j = 1$ we have $a_j^n = 0$ and by definition $f_n(0) = 0 = f_{n+1}(0)$. Suppose we have (2) for some $j$, and we wish to prove it for $j + 1$. Let's compare $f_n$ to $f_{n+1}$ at the right endpoint $b_j^n$ of $I_j^n$.

$$f_n(b_j^n) = f_n(a_j^n) + 3^{-n} \cdot (3/2)^n = f_n(a_j^n) + 2^{-n}. \tag{3}$$

On the other hand $I_j^n$ is the union of three subintervals $[a_i^{n+1}, b_i^{n+1}]$, $[b_i^{n+1}, a_{i+1}^{n+1}]$, $[a_{i+1}^{n+1}, b_{i+1}^{n+1}]$ for some index $i$. By applying (3) to $f_{n+1}$ on the interval $[a_i^{n+1}, b_i^{n+1}]$ we find that

$$f_{n+1}(b_i^{n+1}) = f_{n+1}(a_i^{n+1}) + 2^{-n-1} = f_n(a_j^n) + 2^{-n-1}.$$

By definition, $f_{n+1}$ is constant on the interval $[b_i^{n+1}, a_{i+1}^{n+1}]$, so $f_{n+1}(a_{i+1}^{n+1}) = f_n(a_j^n) + 2^{-n-1}$. By invoking (3) again we find that

$$\begin{aligned}f_{n+1}(b_j^n) = f_{n+1}(b_{i+1}^{n+1}) &= f_{n+1}(a_{i+1}^{n+1}) + 2^{-n-1} \\ &= f_n(a_j^n) + 2 \cdot 2^{-n-1} = f_n(a_j^n) + 2^{-n} = f_n(b_j^n).\end{aligned} \tag{4}$$

Now both $f_n, f_{n+1}$ are constant on the interval $[b_j^n, a_{j+1}^n]$, so we conclude that $f_{n+1}(a_{j+1}^n) = f_n(a_{j+1}^n)$, completing the inductive step. □

(ii) Show that $f$ is constant on each complementary interval $[b_j^k, a_{j+1}^k]$. Conclude that $f([0, 1]\setminus\mathcal{C})$ is countable, hence has measure zero.

**Solution.** That $f$ is constant on each of the indicated intervals is a direct consequence of its definition. Since $[0, 1] \setminus \mathcal{C}$ is the union of all these intervals, and since there are only countably many of them, it follows that $f([0, 1] \setminus \mathcal{C})$ is indeed countable, hence has measure zero. □

(iii) Observe that $f(0) = 0$ and $f(1) = 1$. Show that $f$ maps the interval $[0, 1]$ onto itself. Conclude that $|f(\mathcal{C})| = 1$, even though $|\mathcal{C}| = 0$.

**Solution.** $f$ is continuous, and maps $0, 1$ to themselves, by (2). By the intermediate value theorem, for any $c \in (0, 1)$, there exists $x \in (0, 1)$ satisfying $f(x) = c$. The construction guarantees that $f([0, 1]) \subset [0, 1]$, so $f$ maps $[0, 1]$ onto itself. We've shown in part (ii) that $f(\mathcal{C})$ equals $[0, 1]$ minus a set of measure zero (in fact it maps $\mathcal{C}$ onto $[0, 1]$), so $|f(\mathcal{C})| = 1$. □

(iv) Conclude that $\mathcal{C}$ is uncountable. (In problem 2.1.20 this was proved in a different way.)
**Solution.** If $\mathcal{C}$ were countable then $f(\mathcal{C})$ would be countable, hence would have measure zero, contradicting part (iii). □

(v) Define $g(x) = f(x) + x$, and note that $g(0) = 0$ and $g(1) = 2$. Show that $g$ is *strictly* increasing, that is, if $0 \le x_1 < x_2 \le 1$ then $g(x_1) < g(x_2)$. Thus $g$ is a homeomorphism from $[0, 1]$ to $[0, 2]$.
**Solution.** Each $f_k$ is nondecreasing by construction, so their limit $f$ must also be nondecreasing. Since the function $x \mapsto x$ is strictly increasing, the sum $g(x) = f(x)+x$ is likewise strictly increasing. □

(vi) Show that $g$ maps $\mathcal{C}$ onto a compact set whose Lebesgue measure equals 1. Thus even a homeomorphism can map a set of measure zero onto a set of positive measure.
**Solution.** It suffices to show that $|g([0, 1] \setminus \mathcal{C})| = 1$, for then $|g(\mathcal{C})| = |[0, 2] \setminus g([0, 1] \setminus \mathcal{C})| = 2 - 1$.

Now $[0, 1] \setminus \mathcal{C}$ is the union over all $k, j$ of all the intervals $(b^k_j, a^k_{j+1})$. On each of these intervals, $f$ is constant, so $g(x) = f(b^k_j) + x$. Thus $g$ maps each of these intervals to an interval of the same length. Since $g$ is injective, the measure of the image of their union equals the sum of the lengths, which is 1 (since $|\mathcal{C}| = 0$ and $[0, 1]$ is the union of $\mathcal{C}$ and all these intervals). □

**VII.B(b)** Prove that if $f$ is measurable, then for any $a \in \mathbb{R}$, $f^{-1}(\{a\}) = \{x : f(x) = a\}$ is measurable. Give an example to show that there also exist nonmeasurable functions that satisfy this conclusion.
**Solution.** The set in question is the complement of the union of the two sets $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$. Each of the latter is measurable by part (a) of this problem, so the set in question is likewise measurable since the measurable sets form an algebra. □

Any example must of course rely on the existence of a nonmeasurable set. Take the measure space $(E, \mathcal{A}, \mu)$ to be $(\mathbb{R}^n, \overline{\mathcal{B}}_{\mathbb{R}^n}, \lambda)$ where $\lambda$ denotes Lebesgue measure. We know that there exists a nonmeasurable subset $\mathcal{E}$ of $(0, \infty)$. Define $f(x) = x$ for all $x \in \mathcal{E}$, $f(x) = -x$ whenever $x > 0$ and $x \notin \mathcal{E}$, and $f(x) = 0$ otherwise. For any $y \ne 0$, $f^{-1}(\{y\})$ is a set containing either 0 or 1 elements, hence is Lebesgue measurable. $f^{-1}(\{0\}) = (-\infty, 0]$ is also measurable. On the other hand, $f^{-1}((0, \infty)) = \mathcal{E}$ is not measurable, so $f$ is not a measurable function. □

**VII.C** Let $\{f_n\}$ be a sequence of measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$. Let $L = \{x : \lim_{n\to\infty} f(x)$ exists and is finite$\}$. Show that $L$ is measurable.
**Solution.** First we show that for any sequence $(h_n)$ of measurable functions, $H(x) = \sup_n h_n(x)$ is a measurable function of $x$. This is easy, since for any $a \in \mathbb{R}$ and any point $x$, $H(x) > a$ if and only if there exists $n$ such that $h_n(x) > a$. Therefore

$$\{x : H(x) > a\} = \cup_n \{x : h_n(x) > a\}.$$

Since any countable union of measurable sets is measurable, and since each set on the right-hand side is measurable by hypothesis, this implies that the set on the left is measurable for any $a \in \mathbb{R}$, and hence $H$ is measurable.

Now we show that for any sequence $(f_n)$ of measurable functions, $F(x) = \limsup_{n\to\infty} f_n(x)$ is a measurable function. Define $F_n(x) = \sup_{k\ge n} f_k(x)$. By definition of lim sup, $F(x) = \inf_n F_n(x)$. By the preceding paragraph, we know that each function $F_n$ is measurable. For any $a \in \mathbb{R}$,

$$\{x : \inf_n F_n(x) \ge a\} = \{x : F_n(x) \ge a \ \forall n\} = \cap_{n=1}^{\infty} \{x : F_n(x) \ge a\}.$$

By problem VII.B, each set on the right is measurable, and the intersection of countably many measurable sets is automatically measurable. Finally, by problem VII.B again, measurability of $\{x : \inf_n F_n(x) \ge a\}$ for every $a \in \mathbb{R}$ is equivalent to measurability of $\inf_n F_n = \limsup_{n\to\infty} f_n$.

Next we claim that $\liminf_{n\to\infty} f_n$ is likewise a measurable function. One method of proof is to imitate the proof above, interchanging "$>$" with "$<$" and "$\le$" with "$\ge$" where appropriate. A second method of proof is to note that $\liminf_{n\to\infty} f_n(x) = -\limsup_{n\to\infty}(-f_n(x))$, and to invoke the result for lim sup together with the fact that $-F$ is measurable if and only if $F$ is measurable. Details are left to the reader.

Final step: By the definitions, $\lim_{n\to\infty} f_n(x)$ exists and is finite if and only if the lim sup and lim inf are both finite and are equal. Define $G$ to be the set of all points $x$ for which both the lim sup and the lim inf are finite. $\{x : \limsup_{n\to\infty} f_n(x) = \pm\infty\}$ is measurable, by problem VII.B, since we've shown the lim sup to be a measurable function. The same applies to the lim inf. Thus the complement of $G$ is the union of two measurable sets, so $G$ is measurable.

Define $h(x) = \limsup_n f_n(x) - \liminf_n f_n(x)$ for all $x \in G$. All that remains is to show that $\{x \in G : h(x) = 0\}$ is a measurable set. We cannot directly apply problem VII.B(b), because $h$ is not defined on the whole space. However, an argument explained in class on Wednesday 3/17 does the job:

$$\{x \in G : h(x) > 0\} = \cup_{p,q\in\mathbb{Q};p+q>0}\Big(\{x : \limsup_n f_n(x) > p\} \cap \{x : -\liminf_n f_n(x) > q\} \cap G\Big), \quad (5)$$

so $\{x \in G : h(x) > 0\}$ is measurable. Likewise $\{x \in G : h(x) < 0\}$ is measurable. Thus $\{x \in G : h(x) = 0\}$ can be expressed as $G$ minus the union of two measurable sets, so it is measurable. □

**VII.D** Prove that for any two measurable functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$, the product $fg = f \cdot g$ is measurable.

**Solution.** Set $h = fg$. First consider $h^{-1}(\{0\})$. A product $f(x)g(x)$ equals zero if and only if at least one factor equals zero, so $h^{-1}(\{0\}) = f^{-1}(\{0\}) \cup g^{-1}(\{0\})$. By hypothesis, this is the union of two measurable sets.

Next $h^{-1}(\{+\infty\})$ is the union of the four sets $\{x : f(x) = +\infty \text{ and } g(x) \in (0, +\infty]\}$, $\{x : f(x) = -\infty \text{ and } -g(x) \in (0, +\infty]\}$, $\{x : g(x) = +\infty \text{ and } f(x) \in (0, +\infty]\}$, $\{x : g(x) = -\infty \text{ and } -f(x) \in (0, +\infty]\}$. Each of these four sets is measurable by problem VII(a), so $h^{-1}(\{+\infty\})$ is measurable. Clearly the same reasoning (invoking again problem VII.B(a)) applies to $h^{-1}(\{-\infty\})$.

Consider now any open interval $(a, +\infty)$ where $a > 0$. Then $\{x : h(x) \in (a, +\infty)\}$ is the union of all positive rational numbers $p, q$ satisfying $pq > a$ of $f^{-1}((p, +\infty)) \cap g^{-1}((q, +\infty))$, together with the union over the same numbers $p, q$ of $f^{-1}((-\infty, -p)) \cap -g^{-1}((-\infty, -q))$. Since $\mathbb{Q}^+ \times \mathbb{Q}^+$ is countable, and any countable subset of a countable set is countable, this is a countable union of measurable sets (by hypothesis together with problem VII.B(a)), hence is measurable.

Now for any $b \in \mathbb{R}$, whether positive, negative, or zero, $h^{-1}(b, +\infty]$ is a finite union of certain of the sets which we have just shown to be measurable. Therefore it is measurable, and hence, by definition, $h = fg$ is measurable. □

**VII.E** Give an example of a measurable function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every $x \in \mathbb{R}$. (You need not give a detailed proof.)

**Solution.** Let $f = \chi_{\mathbb{Q}}$, that is, $f(x) = 1$ for all $x \in \mathbb{Q}$, and $f(x) = 0$ for all $x \notin \mathbb{Q}$. □