

Mathematics 105, Spring 2004 — Problem Set V Solutions¹

V.A Show that any countable subset of \mathbb{R}^n is measurable, and has measure zero.

Solution. First of all, a set containing a single point y is measurable since it is closed, and has measure zero since it is a closed rectangle of volume zero. (If you prefer to consider only rectangles having strictly positive volumes to be rectangles, then you can instead note that $\{y\}$ is covered by the single rectangle $[y_1 - \varepsilon, y_1 + \varepsilon] \times \cdots \times [y_n - \varepsilon, y_n + \varepsilon]$, which has volume $2^n \varepsilon^n$, and this tends to zero as $\varepsilon \rightarrow 0$.)

Now any countable set E is a countable union of sets each of which contains a single point, hence is a countable union of sets E_j of measure zero, and hence $|E| \leq \sum_j |E_j| = \sum_j 0 = 0$. \square

V.B Let $A_1 \supset A_2 \supset A_3 \supset \cdots$ be an infinite sequence of nested measurable subsets of \mathbb{R}^n . Let A be their intersection. Show that if $|A_1| < \infty$ then $|A| = \lim_{n \rightarrow \infty} |A_n|$.

Solution. First of all note that it suffices to prove this in the special case where $A = \emptyset$. Indeed, in the general case define $\tilde{A}_j = A_j \setminus A$. Then the sets \tilde{A}_j are still measurable (since the measurable sets form a σ -algebra), and their intersection is empty. From the special case we therefore conclude that $|\tilde{A}_j| \rightarrow 0$ as $j \rightarrow \infty$. Now A_j is the disjoint union $\tilde{A}_j \cup A$, so $|A_j| = |A| + |\tilde{A}_j| \rightarrow |A|$.

Thus we may suppose henceforth that $\cap_j A_j = \emptyset$. Define auxiliary sets B_j by $B_j = A_j \setminus A_{j+1}$ for all $j \geq 1$. Then $A_1 = \cup_{j=1}^{\infty} B_j$, and the sets B_j are pairwise disjoint because the sets A_j are nested. Since each A_j is measurable, so is $\mathbb{R}^n \setminus A_j$, and hence so is $B_j = A_j \cap (\mathbb{R}^n \setminus A_{j+1})$.

Since the sets B_j are measurable and pairwise disjoint, $|\cup_{j=1}^{\infty} B_j| = \sum_{j=1}^{\infty} |B_j|$. This union equals A_1 , which has finite measure. We conclude that the infinite series $\sum_j |B_j|$ converges. Thus $\lim_{N \rightarrow \infty} \sum_{j \geq N} |B_j| \rightarrow 0$ as $N \rightarrow \infty$. But $A_N = \cup_{j \geq N} B_j$, so $|A_N| \leq \sum_{j \geq N} |B_j| \rightarrow 0$. \square

Comment. Note how the *countable* additivity of Lebesgue measure is the key to evaluating the limit. This connection between countable additivity and various limiting operations is an essential feature of the theory, and is what makes Lebesgue measure and integration more successful than the older theory of Jordan content.

Comment. The conclusion doesn't hold in general without the hypothesis that A_1 has finite measure. Consider for instance the case where $A_n = [n, \infty)$. Then $|A_n| = \infty$ for all n , yet $\cup_n A_n = \emptyset$.

V.C Fix a parameter $\alpha \in (0, 1)$. Begin with $E_0 = [0, 1]$. Define E_1 to be the set obtained by deleting from E_0 an interval of length $\alpha/3$ centered at $\frac{1}{2}$. Thus E_1 is the union of two closed intervals, I_1^1 and I_2^1 , each of which has length $\frac{1}{2}(1 - \frac{\alpha}{3})$. From each of those two intervals I_j^1 delete an open interval of length $\alpha/3^2$ whose center is the center of I_j^1 , and let E_2 be the resulting subset of E_1 ; thus E_2 is a union of four closed intervals I_j^2 , each having length $\frac{1}{4}(1 - \frac{\alpha}{3} - \frac{2\alpha}{3^2})$. At the next stage delete an interval of length $\alpha/3^3$ from the center of each of the four intervals I_j^2 . Continue this process indefinitely, obtaining an infinite sequence of sets $E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots$. Each set E_n is the union of 2^n intervals I_j^n , each having length $2^{-n}(1 - \frac{\alpha}{3} - \cdots - \frac{2^{n-1}\alpha}{3^n})$. Define $E = \cap_{n=0}^{\infty} E_n$. (Informally, since

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these sets are nested, E is the limit of E_n as $n \rightarrow \infty$.) (a) Show that E is compact and that $|E| = 1 - \alpha$.

Solution. Each set E_n is a finite union of closed intervals, hence is closed. The intersection of any family of closed sets is closed, and $E \subset [0, 1]$, so E is bounded and thus compact. By problem V.B, $|E| = \lim_{n \rightarrow \infty} |E_n|$. Since E_n is a disjoint union of 2^n intervals, its measure is the sum of their lengths, by one of our first lemmas. Thus $|E_n| = 2^n \cdot 2^{-n}(1 - \frac{\alpha}{3} - \frac{2\alpha}{3^2} - \dots - \frac{2^{n-1}\alpha}{3^n})$. As $n \rightarrow \infty$ this tends to $1 - \frac{1}{3}\alpha \sum_{k=0}^{\infty} (2/3)^k = 1 - \alpha$. \square

(b) Show that the interior of E is empty. (Equivalently, E contains no interval of positive length.)

Solution. Any interval contained in E is contained in all E_n , and it is apparent that E_n contains no interval of length larger than $2^{-n}(1 - \frac{\alpha}{3} - \frac{2\alpha}{3^2} - \dots - \frac{2^{n-1}\alpha}{3^n})$. This quantity is $\leq 2^{-n}$, which tends to zero as $n \rightarrow \infty$. \square

V.D Express any $x \in [0, 1]$ in base two notation, that is, $x = \sum_{k=1}^{\infty} a_k(x)2^{-k}$ where $a_k \in \{0, 1\}$. If x has two different binary expansions then we choose the one which does not end in an infinite string of zeros. Let E be the set of all numbers whose binary expansion does not contain any two consecutive zeros (that is, if $a_k(x) = 0$ then $a_{k+1}(x) = 1$). Prove that E is measurable, and that $|E| = 0$.

Solution. In order to show that a set has measure zero, it suffices to find some larger set whose measure is zero. This can be advantageous if one is dealing with a complicated set; such a set might be contained in a simpler set which is larger, but still small enough to have measure zero. That's what I'll do here.

Define the simpler set \tilde{E} to consist of all $x = \sum_{k=1}^{\infty} a_k 2^{-k}$, where $a_k = a_k(x)$ as in the problem statement, with the following property: for any $j \in \mathbb{N}$, a_{2j-1}, a_{2j} are not both zero. (Points in the set E also satisfy additional constraints; for instance, a_2 and a_3 can't both be zero.) Clearly $E \subset \tilde{E}$, so if we show that $|\tilde{E}|_e = 0$ then we have $|E|_e = 0$ and hence E is measurable and $|E| = 0$.

To analyze \tilde{E} it helps to note that it is quite similar to the Cantor set discussed in Stroock's problem 2.1.20. Define the approximating sets \tilde{E}_n as follows: \tilde{E}_n is the set of all $x = \sum_{k=1}^{\infty} a_k 2^{-k}$ such that for all $1 \leq j \leq n$, a_{2j-1}, a_{2j} are not both zero. Since $\tilde{E} \subset \tilde{E}_n$, it suffices to show that $|\tilde{E}_n|_e \rightarrow 0$ as $n \rightarrow \infty$.

Now $\tilde{E}_1 = [0, 1] \setminus [0, \frac{1}{4}) = [\frac{3}{4}, 1]$; we've deleted the leftmost quarter of $[0, 1]$. This is the union of 3 intervals, each of length $\frac{1}{4}$. A brief inspection shows that \tilde{E}_2 is the union of $9 = 3^2$ intervals, each of length 4^{-2} ; one deletes the leftmost quarter of each of the 3 intervals out of which \tilde{E}_1 is built. Similarly each \tilde{E}_n is a union of 3^n intervals, each of length 4^{-n} . Thus $|\tilde{E}_n| = (3/4)^n$, which certainly tends to zero. \square

Alternative Solution. One could work directly with the given set E , but the argument becomes more complicated. Here's an outline: (i) Define g_n to be the number of finite sequences (a_1, a_2, \dots, a_n) for each $a_k \in \{0, 1\}$, and no two consecutive numbers a_k, a_{k+1} are both zero. Then $g_{n+1} = g_n + g_{n-1}$. Indeed a sequence $(a_1, \dots, a_n, a_{n+1})$ satisfies the given constraint that no two consecutive numbers vanish if and only if either $a_{n+1} = 1$ and (a_1, \dots, a_n) satisfies the constraint, or $a_{n+1} = 0$, $a_n = 1$, and (a_1, \dots, a_{n-1}) satisfies the constraint. These possibilities are mutually exclusive, so the total number of possibilities is $g_n + g_{n-1}$. (ii) Recall the famous Fibonacci sequence $(F_0, F_1, F_2, \dots) = (1, 1, 2, 3, 5, \dots)$, which likewise satisfies the recursion $F_{n+1} = F_n + F_{n-1}$. Since $g_1 = F_2$ and $g_2 = F_3$, the recursion relation implies that $g_n \equiv F_{n+1}$ for all n . (iii) A basic

fact about the Fibonacci sequence is that it is intimately connected with the golden ratio $\gamma = \frac{1+\sqrt{5}}{2}$; $\lim_{n \rightarrow \infty} F_n/\gamma^n$ exists and equals a positive number (whose exact value doesn't matter here). **(iv)** Define E_n to be the set of all $x \in [0, 1]$ for which the first n digits $(a_k(x) : 1 \leq k \leq n)$ satisfy the constraint that no two consecutive digits vanish. Then E_n is the union of g_n intervals, each having length 2^{-n} ; these intervals are nonoverlapping. Therefore $|E_n| = g_n 2^{-n} = F_{n+1} 2^{-n}$. **(v)** By problem V.B, $|E| = \lim_{n \rightarrow \infty} |E_n|$. Since $F_{n+1} \gamma^{-n}$ converges to a finite limit, and $\gamma^n 2^{-n} \rightarrow 0$ since $\gamma < 2$, $|E_n| \rightarrow 0$. \square

V.E(a) Let $E \subset \mathbb{R}^n$ be a measurable set. Show that for *any* set $A \subset \mathbb{R}^n$, $|A|_e = |A \cap E|_e + |A \setminus E|_e$.

Solution. Of course we have $|A|_e \leq |A \cap E|_e + |A \setminus E|_e$ for any two sets A, E , since $A = (A \cap E) \cup (A \setminus E)$, and outer measure is subadditive.

To prove the converse inequality, let $\varepsilon > 0$ be arbitrary. There exists an open set $G \supset A$ satisfying² $|G| \leq |A|_e + \varepsilon$. Since G, E are both measurable, $|G| = |G \cap E| + |G \setminus E|$. Thus

$$|A|_e \geq |G| - \varepsilon = |G \cap E| + |G \setminus E| - \varepsilon \geq |A \cap E|_e + |A \setminus E|_e - \varepsilon;$$

the last inequality holds because $G \cap E \supset A \cap E$ and likewise $G \setminus E \supset A \setminus E$. Since this holds for arbitrary $\varepsilon > 0$, it follows that $|A|_e \geq |A \cap E|_e + |A \setminus E|_e$. \square

Comment. It is possible to prove this in various other ways. **(i)** ε can be eliminated, by using the fact that there exists a \mathfrak{G}_δ set $\tilde{G} \supset A$ satisfying $|\tilde{G}| = |A|_e$. **(ii)** Instead of approximating A by open sets, one can approximate E by open and compact sets, from the inside and outside respectively, and use the fact that outer measure is additive for sets separated by a positive distance. This leads to a more complicated proof, though.

(b) Show conversely that if E is a set for which the above identity holds for all sets A , then E must be measurable.

Solution. First suppose that $|E|_e < \infty$. Let $\varepsilon > 0$. Choose an open set $G \supset E$ such that $|G| < |E|_e + \varepsilon$. Then $|G| = |G \cap E|_e + |G \setminus E|_e = |E|_e + |G \setminus E|_e$, so $|G \setminus E|_e \leq |G| - |E|_e < \varepsilon$. Therefore E satisfies the definition of measurability.

Now consider the general case. Note that if E satisfies the hypothesis, then for any open ball B and any set $A \subset B$,

$$|A \cap (E \cap B)|_e + |A \setminus (E \cap B)|_e = |A \cap E|_e + |A \setminus E|_e = |A|_e. \quad (1)$$

Using this plus the fact that $|E \cap B|_e \leq |B| < \infty$, we can prove that $E \cap B$ is measurable: Let $\varepsilon > 0$, choose an open set $G \supset E \cap B$ satisfying $|G| < |E \cap B|_e + \varepsilon$, and replace G by $\tilde{G} = G \cap B$, which is an open subset of B , contains $E \cap B$, and of course satisfies $|\tilde{G}| < |E \cap B|_e + \varepsilon$. Then just as in the preceding paragraph, it follows from (1) that $|\tilde{G} \setminus (E \cap B)|_e < \varepsilon$.

The last step is to note that $E = \cup_{j=1}^{\infty} [E \cap B(0, j)]$ is a countable union of measurable sets, hence is measurable. \square

Problem set VI. Read §§2.2,3.1 of Stroock. (In §3.1, don't pay close attention to the discussion of π - and λ -systems, but do study the rest.) Solve problems 2.2.3, 2.2.4, 3.1.9, 3.1.11, 3.1.12 of Stroock. (Note typo in 2.2.3(ii): The first part of the displayed equation should read $|B_{\mathbb{R}^N}(c, r)| = \dots$)

²As was stressed Wednesday 3/3 in class, one can't say that $|G| < |A|_e + \varepsilon$, because $|A|_e$ could be infinite.