Mathematics 105, Spring 2004 — Problem Set IV Solutions [1]

IV.A Let $\{I_n\}$ be any *finite* set of open intervals that covers $[0,1]\cap\mathbb{Q}$. Show that $\sum_n |I_n| \ge 1$. Explain why this does *not* prove that $|[0,1]\cap\mathbb{Q}|_e \ge 1$.

Solution. This does not prove that $|[0,1]\cap\mathbb{Q}|_e \ge 1$, because in the definition of outer measure, *infinite* covers are also allowed. This problem shows that there is a dramatic difference between outer measure and the variant that would be obtained by allowing only finite covers.

Suppose that $\{I_n : 1 \le n \le N\}$ is some open cover of $[0,1]\cap\mathbb{Q}$ by finitely many intervals. We claim more generally that any finite cover of $(A,B)\cap\mathbb{Q}$ by intervals I_n must satisfy $\sum_n |I_n| \ge B-A$. We prove this by induction on N, the case $N=1$ being obvious. Clearly there must exist some index k such that either $A \in I_k$, or A is the left endpoint of I_k. Let b be the right endpoint of I_k. If $b > B$ then already $|I_k| \ge B-A$, and we're done. If $b < B$ then $\{I_n : n \ne k\}$ is a cover of $(b,B)\cap\mathbb{Q}$ by $N-1$ intervals, so by induction $\sum_{n\ne k}|I_n| \ge 1-b$. Thus $\sum_n |I_n| \ge (B-b)+|I_k| \ge (B-b)+(b-A) = B-A$. □

IV.B Show that for any Lebesgue measurable sets $A, B \subset \mathbb{R}^n$, $|A\cup B| + |A\cap B| = |A| + |B|$.

Solution. Decompose $A\cup B = (A\setminus B)\cup(A\cap B)\cup(B\setminus A)$; the three sets on the right-hand side are pairwise disjoint. Since $\overline{\mathcal{B}}$ is a σ-algebra, all three of these sets are measurable. Therefore $|A\cup B| = |A\setminus B| + |A\cap B| + |B\setminus A|$ and consequently the left-hand side of our inequality is $|A\cup B| = |A\setminus B| + 2|A\cap B| + |B\setminus A|$. Now $A = (A\setminus B)\cup(A\cap B)$ and the two sets on the right are disjoint, so $|A| = |A\setminus B| + |A\cap B|$. Likewise $|B| = |B\setminus A| + |A\cap B|$. Thus the inequality holds. □

Comment. It's possible that various of these sets have infinite measures. The sum of two elements of $[0,+\infty]$ is defined to be $+\infty$ if one or both of the summands are infinite, and to be the usual sum of both are finite. With this convention, which we use throughout the course, the above manipulations are valid.

Alternatively, one can minimize manipulation of infinite quantities by splitting the proof into two cases. If either of A, B have infinite measure then so does $A\cup B$, and hence both sides of the inequality equal $+\infty$ and are therefore equal. Otherwise all the sets appearing in the above analysis have finite measures. □

IV.C Let $A \subset \mathbb{R}^n$ and suppose that $|A|_e = 0$. Prove that for any set $B \subset \mathbb{R}^n$, $|A\cup B|_e = |B|_e$.

Solution. Certainly $|A\cup B|_e \ge |B|_e$ since $A\cup B \supset B$. Thus it suffices to prove the converse inequality. Now $|A\cup B|_e \le |A|_e + |B|_e$ for any sets A, B. In the present problem $|A|_e = 0$, so we obtain $|A\cup B|_e \le |B|_e$, which is the desired converse inequality. □

IV.E Let the sequence $(x_n)_{n=1}^\infty$ be an enumeration of $[0,1]\cap\mathbb{Q}$. (That is, for each $q \in [0,1]\cap\mathbb{Q}$ there exists a unique $n \in \mathbb{N}$ such that $q = x_n$.) Define the open intervals $I_n = (x_n - \delta_n, x_n + \delta_n)$ where $\delta_n = \frac{1}{10}2^{-n}$.

Let $E = \cup_{n=1}^\infty I_n$. (a) Show that E is a dense subset of $[0,1]$.

Solution. It is a basic fact (Math 104) that any open subinterval of $\mathbb{R}$ contains at least one (in fact, infinitely many) rational numbers. Since E contains every rational number in $[0,1]$, E is therefore dense in $[0,1]$. □

(b) Show that $E \ne [0,1]$. (Hint: Find an upper bound for $|E|$.)

Solution. E is open, hence measurable. $|E| = |\cup_n I_n| \le \sum_n |I_n| = \sum_{n=1}^\infty \frac{1}{5}2^{-n} = \frac{1}{5}$. (This is certainly an overestimate, since many of the intervals I_n must overlap one another; many will be completely contained in others.) Thus $|E| < |[0,1]|$, so E cannot contain all of $[0,1]$. □

Comment. This example shows that open sets, even of $\mathbb{R}^1$, can be quite a bit more complicated than those we usually draw. By replacing the factor $\frac{1}{10}$ by a sufficiently small number we can in the same way create an open set which is dense in $[0,1]$, yet has arbitrarily small measure. This problem gives us an example of an open set E whose closure has strictly larger measure than E; equivalently, the boundary of E has strictly positive measure!

Problem Set V: For Friday March 5, solve problem 2.1.20 of Stroock, problem IV.D, and the following ones. (See below for a comment on # 2.1.20.)

V.A Show that any countable subset of $\mathbb{R}^n$ is measurable, and has measure zero.

V.B Let $A_1 \supset A_2 \supset A_3 \supset \cdots$ be an infinite sequence of nested measurable subsets of $\mathbb{R}^n$. Let A be their intersection. Show that if $|A_1| < \infty$ then $|A| = \lim_{n\to\infty} |A_n|$.

V.C In problem 2.1.20 we constructed an uncountable set of measure zero. In this problem we discuss a variant of this construction which produces a compact subset of $\mathbb{R}^1$ which has positive measure, yet which does not contain any interval of positive length.

Fix a parameter $\alpha \in (0,1)$. Begin with $E_0 = [0,1]$. Define E_1 to be the set obtained by deleting from E_0 an interval of length $\alpha/3$ centered at $\frac{1}{2}$. Thus E_1 is the union of two closed intervals, I_1^1 and I_2^1, each of which has length $\frac{1}{2}(1-\frac{\alpha}{3})$. From each of those two intervals I_j^1 delete an open interval of length $\alpha/3^2$ whose center is the center of I_j^1, and let E_2 be the resulting subset of E_1; thus E_2 is a union of four closed intervals I_j^2, each having length $\frac{1}{4}(1-\frac{\alpha}{3}-\frac{2\alpha}{3^2})$. At the next stage delete an interval of length $\alpha/3^3$ from the center of each of the four intervals I_j^2. Continue this process indefinitely, obtaining an infinite sequence of sets $E_0 \supset E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots$. Each set E_n is the union of 2^n intervals I_j^n, each having length $2^{-n}(1-\frac{\alpha}{3}-\cdots-\frac{2^{n-1}\alpha}{3^n})$. Define $E = \cap_{n=0}^{\infty} E_n$. (Informally, since these sets are nested, E is the limit of E_n as $n \to \infty$.)

(a) Show that E is compact and that $|E| = 1-\alpha$. (b) Show that the interior of E is empty. (Equivalently, E contains no interval of positive length.)

V.D Imagine picking a number x in $[0,1]$ at random. What could this mean? Let $E \subset [0,1]$. We *define* the *probability that a randomly chosen number x belongs to* E to be $|E|$, provided that E is measurable; otherwise the probability is undefined.

Imagine expressing x in base two notation; that is, $x = \sum_{k=1}^{\infty} a_k(x)2^{-k}$ where $a_k \in \{0,1\}$, and if x has two different binary expansions then we choose the one which does not end in an infinite string of zeros. (Thus $\frac{1}{4}$ is expressed as $\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots$, and so on; $a_k(\frac{1}{4}) = 0$ for $k = 1,2$ and $= 1$ for all $k \geq 3$.)

Let E be the set of all numbers whose binary expansion does not contain any two consecutive zeros (that is, if $a_k(x) = 0$ then $a_{k+1}(x) = 1$). Prove that E is measurable, and that the probability that a randomly chosen number belongs to E is zero. (This problem can be restated without using the words "probability" and "random" at all; you're just asked to show that a particular set has measure zero.)

V.E Let $E \subset \mathbb{R}^n$ be a measurable set. (a) Show that for *any* set $A \subset \mathbb{R}^n$, $|A|_e = |A \cap E|_e + |A \setminus E|_e$. (Warning: This is not obvious.)

(b) Show conversely that if E is a set for which the above identity holds for all sets A, then E must be measurable. (Comment: In some treatments this property is taken as the definition of a measurable set. I don't like this because the motivation is opaque, whereas the definition in our text has a clear heuristic interpretation: The measurable sets are the ones which are closely approximable by open sets, in a certain precise sense.)

Comment on #2.1.20: At the very end, Stroock refers to Cantor's anti-diagonalization procedure for showing that $\{0,2\}^{\mathbb{N}}$ cannot be counted. (Here $\{0,2\}^{\mathbb{N}}$ is by definition the set of all sequences $x = (x_1, x_2, \cdots)$ for which each x_j belongs to $\{0,2\}$.) That procedure is as follows: Suppose that we were given an enumeration $y_1, y_2, \cdots$ of $0,2^{\mathbb{N}}$; this means that each y_n belongs to $0,2^{\mathbb{N}}$, and every element of $0,2^{\mathbb{N}}$ equals y_n for at least one value of n. (Each y_n is itself an infinite sequence of zeros and twos.) Write $y_n = (y_{n,1}, y_{n,2}, y_{n,3}, \cdots)$. Define an element $x = (x_1, x_2, \cdots) \in 0,2^{\mathbb{N}}$ by $x_k = 0$ if $y_{k,k} = 2$, and $x_k = 2$ if $y_{k,k} = 0$. Then for any n, $x \neq y_n$ since the n-th component x_n of x does not equal the n-th component $y_{n,n}$ of y_n. Thus the sequence $y_1, y_2, \cdots$ cannot be an enumeration of all of $0,2^{\mathbb{N}}$.