Mathematics 105 — Spring 2004 — M. Christ[1]

Midterm Exam — Wednesday February 18

Guidelines. The exam will be based on the first three problem sets, on Chapters 1 and 2 of Spivak, and on the lectures. You might be asked to solve problems similar to those in the problem sets, to state definitions, to state theorems, and to give examples (possibly with justification). Mastery of the proofs in the text, especially those discussed in lecture, could be tested; you might be asked to reproduce a proof, or to carry out one step in a proof, or to use theorems, lemmas, and arguments from the text to prove other statements.

Solutions to selecta from problem set #3

2-29(b). Show that if the directional derivative $D_x f(a)$ exists, then for any $t \in \mathbb{R}$, $D_{xt} f(a) = tD_x f(a)$.

Solution. We are given that $\lim_{s\to 0} \frac{f(a+sx)-f(a)}{s}$ exists, and we are asked to evaluate $\lim_{h\to 0} \frac{f(a+htx)-f(a)}{h}$. If $t = 0$ then the latter limit is $\lim_{h\to 0} \frac{0}{h} = 0 = 0 \cdot D_x f(a)$, so the desired equation holds. If $t \neq 0$ we evaluate the limit by making the substitution $h = s/t$, where t remains fixed and s is a function of h. Certainly $s \to 0$ as $h \to 0$ and vice versa, so we obtain

$$\lim_{h\to 0} \frac{f(a+htx)-f(a)}{h} = \lim_{s\to 0} \frac{f(a+sx)-f(a)}{s/t} = t \lim_{s\to 0} \frac{f(a+sx)-f(a)}{s} = tD_x f(a).$$

Thus $D_{tx} f(a)$ exists and equals $tD_x f(a)$. □

2-29(c). Suppose that f is differentiable at a. Show that $D_x f(a)$ exists and equals $Df(a)(x)$, and that $D_{x+y} f(a) = D_x f(a) + D_y f(a)$.

Solution. The latter conclusion follows from the former, because $Df(a)(x) + Df(a)(y) = Df(a)(x+y)$ since $Df(a)$ is a linear transformation.

If $x = 0$ then $D_x f(a) = 0$, as one sees directly from its definition (see the solution for part (b)), while $Df(a)(x) = 0$ as well, so they are equal.

Assume now that $x \neq 0$. We know from the definition of $Df(a)$ that $\frac{|f(a+tx)-f(a)-Df(a)(tx)|}{|tx|}$ tends to zero as $t \to 0$. Pulling out the nonvanishing factor of $|x|$ from the denominator doesn't change this, and a little algebraic manipulation allows us to rewrite this fact as

$$\lim_{t\to 0} \left| \frac{f(a+tx)-f(a)}{t} - Df(a)(x) \right| = 0.$$

This means precisely that $\frac{f(a+tx)-f(a)}{t} \to Df(a)(x)$ as $t \to 0$. Thus $D_x f(a)$ exists and equals $Df(a)(x)$. □

2-35. Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be a continuously differentiable function, satisfying $f(0) = 0$. Find functions $g_j : \mathbb{R}^n \to \mathbb{R}$ satisfying $f(x) = \sum_{j=1}^n x^j g_j(x)$ for all $x \in \mathbb{R}^n$. (Recall Spivak's notation $x = (x^1, \cdots, x^n)$; the superscripts indicate components of a vector, rather than exponents.)

Comment. This is an underdetermined problem if $n > 1$; we have one equation and n unknowns. There are zillions of solutions, but the one we'll obtain has the advantage that if f is for instance twice continuously differentiable, then the functions g_j turn out to be differentiable; this is useful in certain applications.

Comment. For $n > 1$, this is a kind of higher-dimensional *division formula*. For $n = 1$ the formula $f = \sum_j x^j g_j(x)$ becomes $f = xg_1(x)$, which is equivalent for $x \neq 0$ to $g_1(x) = f(x)/x$.

Solution. For each $x \in \mathbb{R}^n$ define $h_x : \mathbb{R}^1 \to \mathbb{R}^1$ by $h_x(t) = f(tx)$. Consider $\int_0^1 h_x'(t)\,dt$. Fix any $x \in \mathbb{R}^n$, and let t vary. Then $h_x'(t)$ is a continuous function of t; we'll verify this below. Granting this, the fundamental theorem of one-variable calculus tells us that

$$\int_0^1 h_x'(t)\,dt = h_x(1) - h_x(0) = f(x) - f(0) = f(x).$$

The trick is to evaluate this same integral in another way. By the chain rule, $h_x'(t) = \sum_{j=1}^n x^j D_j f(tx)$. This shows that $h_x'(t)$ is indeed a continuous function of t. Now we insert this into the integral:

$$\int_0^1 h_x'(t)\,dt = \int_0^1 \sum_{j=1}^n x^j D_j f(tx)\,dt = \sum_{j=1}^n x^j \int_0^1 D_j f(tx)\,dt.$$

Defining $g_j(x) = \int_0^1 D_j f(tx)\,dt$, we therefore have $f(x) = \sum_{j=1}^n x^j g_j(x)$ for all $x \in \mathbb{R}^n$. □

Comment. Here's another solution: First define $g_j(0) = 0$ for all j; then the desired equation holds for $x = 0$. Consider any $0 \neq x \in \mathbb{R}^n$. Define $k(x)$ to be the smallest index $j \in \{1, 2, \cdots, n\}$ satisfying $x^j \neq 0$. Since $x \neq 0$, $k(x)$ exists and is well-defined. Define $g_i(x) = 0$ if $i \neq k(x)$, and $g_i(x) = f(x)/x_i$ if $i = k(x)$. These functions g satisfy the desired relation. □

2-37(a). Let $f : \mathbb{R}^2 \to \mathbb{R}^1$ be continuously differentiable. Show that f is not one-to-one.

Solution. We'll prove more: In any open set, no matter how small, f fails to be injective. Let B be any open ball in $\mathbb{R}^2$. The proof is by cases. The first case is when $D_1 f(x, y) = D_2 f(x, y) = 0$ for every point $(x, y) \in B$. Then f is constant in B, as we showed in a problem earlier in the course.

The second case is when there exists $z = (x_0, y_0) \in B$ satisfying $D_1 f(z) \neq 0$. The third case is when instead $D_2 f(z) \neq 0$ for some $z \in B$. By reversing the roles of the two coordinates we see that the third case follows from the second (that is, define $g(x, y) = f(y, x)$; if $D_2 f(x_0, y_0) \neq 0$ then $D_1 g(y_0, x_0) \neq 0$).

So we may assume that $D_1 f(z) \neq 0$. By the implicit function theorem applied to $F(x, y) = f(x, y) - f(z)$, there exists a continuously differentiable function h, defined in some open interval I containing y_0, satisfying $h(y_0) = x_0$ and $F(h(y), y) = 0$, for all $y \in I$. Thus $f(h(y), y) = f(x_0, y_0)$ for all $y \in I$. Since h is continuous, the point $(h(y), y)$ belongs to B for all y sufficiently close to y_0. And clearly if $y \neq \tilde{y}$ then $(h(y), y) \neq (h(\tilde{y}), \tilde{y})$. Thus f is not injective in B. □

Comment. In our text this problem precedes the implicit function theorem, so you can complain that it's not kosher for me to invoke that theorem, and I agree. But observe that all our author has done in the hint is to reproduce the device used in the proof of the implicit function theorem to reduce it to the inverse function theorem. We could also solve this problem by following the hint, and invoking the inverse function theorem, as follows:

In the case where $D_1 f(z) \neq 0$, consider $G(x, y) = (f(x, y), y)$. The Jacobian matrix is $G'(z) = \begin{pmatrix} D_1 f(z) & D_2 f(z) \\ 0 & 1 \end{pmatrix}$, so $G'(z)$ is nonsingular and the inverse function theorem applies. Therefore there exists a function H which maps some open set containing $(f(z), y_0)$ to some open set containing $z = (x_0, y_0)$, and which inverts G on these sets.

In all equations I'm about to write, it is assumed that (x, y) is sufficiently close to z that $G(x, y)$ belongs to the domain of H. Thus $(x, y) = H(f(x, y), y)$. Writing $H = (H^1, H^2)$ we have $x = H^1(f(x, y), y)$, and $y = H^2(f(x, y), y)$. Define $h(y) = H^1(f(z), y)$. Then $(h(y), y) = H(f(z), y)$, so $G((h(y), y)) = G \circ H((f(z), y)) = (f(z), y)$. By the definition of G this means that $f(h(y), y) = f(z)$. Thus we've found a whole curve of points, those of the form $(h(y), y)$ for y sufficiently close to y_0, which are mapped by f into the same number, $f(z)$. Thus f is not injective in any open set containing z. $\square$

2-41(c). Let $f(x, y) = x(y \log y - y) - y \log x$. Find $\max_{x \in [\frac{1}{2}, 2]} \min_{y \in [\frac{1}{3}, 1]} f(x, y)$. (Here log denotes the natural logarithm.)

Comment. In parts (a) and (b) Spivak intends us to assume that f is twice continuously differentiable. In part (b), for instance, he is clearly assuming at a minimum that the second partial derivative $D_{2,1} f$ exists.

Solution. For any $x \in [\frac{1}{2}, 2]$, define $g_x(y) = f(x, y)$ and $h(x) = \min_{y \in [\frac{1}{3}, 1]} f(x, y)$. Then h is continuous; I leave it to you to verify this. Therefore h is a bounded function of $x \in [\frac{1}{2}, 2]$, and attains its maximum value M at some point of this interval.

Consider the subproblem of minimizing $g_x(y)$ over the interval $[\frac{1}{3}, 1]$. $g'_x(y) = D_2 f(x, y) = x \log y - \log x$. There exists a unique $y = c(x) \in \mathbb{R}^+$ for which $D_2 f(x, y) = 0$, namely $y = c(x) = x^{1/x} = \exp(\log(x)/x)$. Moreover $D_{2,2} f(x, y) = x/y$ is everywhere positive (for $x, y > 0$). Since $g''_x(y) = D_{2,2} f(x, y) > 0$, g_x attains its minimum value, over all $y > 0$, at $c(x)$. Matters are complicated by the fact that $c(x)$ need not lie in the prescribed interval $[\frac{1}{3}, 1]$. We know that $h(x) = f(x, c(x))$ if $c(x) \in [\frac{1}{3}, 1]$, and otherwise $h(x) = \min(f(x, \frac{1}{3}), f(x, 1))$.

Suppose that there exists some point (x_0, y_0) in the open rectangle $(\frac{1}{2}, 2) \times (\frac{1}{3}, 1)$ such that $f(x_0, y_0) = M$ and $y_0 = c(x_0)$. Since $c(x_0) \in (\frac{1}{3}, 1)$, likewise $c(x) \in (\frac{1}{3}, 1)$ for all x sufficiently close to x_0, so $h(x) = f(x, c(x))$ for all x in some open interval containing x_0. Since $y_0 = c(x_0)$, $D_2 f(x_0, y_0) = 0$. Moreover, since h has a local maximum at x_0, $h'(x_0)$ must equal 0. On the other hand applying the chain rule to the formula $h(x) = f(x, c(x))$ yields

$$h'(x) = D_1 f(x, c(x)) + D_2 f(x, c(x)) \cdot c'(x).$$

Since $D_2 f(x, c(x)) = 0$,

$$h'(x) = D_1 f(x, c(x)) = c(x) \log c(x) - c(x) - c(x) x^{-1} = x^{1/x} x^{-1} (\log x - x - 1).$$

Each term here is negative for all $x \in [\frac{1}{2}, 2]$, so $h'(x_0)$ cannot vanish. Thus we find that h cannot attain its maximum at any point $(x, c(x))$ with $c(x) \in (\frac{1}{3}, 1)$ and $x \in (\frac{1}{2}, 2)$.

Matters now reduce to finding the maximum over all $x \in [\frac{1}{2}, 2]$ of $\min(f(x, \frac{1}{3}), f(x, 1))$. Both $f(x, \frac{1}{3})$ and $f(x, 1)$ are decreasing functions of $x \in [\frac{1}{2}, 2]$, as one sees by taking their derivatives. Thus $M = \min(f(\frac{1}{2}, \frac{1}{3}), f(\frac{1}{2}, 1))$.

Now the function $\varphi(y) = f(\frac{1}{2}, y)$ satisfies $\varphi'(y) = \frac{1}{2}\log y + \log 2$, which is positive for all $y \in [\frac{1}{3}, 1]$ as one can see by noting that its derivative is positive, and computing its value at $y = \frac{1}{3}$. Thus it attains its minimum at $y = \frac{1}{3}$. Thus the final answer is

$$M = f(\tfrac{1}{2}, \tfrac{1}{3}) = -\tfrac{1}{6}\log(3) - \tfrac{1}{6} + \tfrac{1}{3}\log(2).$$

□

Comment. It's also possible to treat this problem by splitting it into three cases: $c(x) \in (\frac{1}{3}, 1)$, $c(x) \leq \frac{1}{3}$, and $c(x) \geq 1$. This then gives rise to three one-variable maximization problems. By solving each, then taking the maximum of the three results, one eventually gets the same answer. But this is a bit more complicated than the solution I've given.

Problem Set 4

There is no problem set due on Friday, February 20; instead a longer one will be due the following Friday. Please do read Stroock §§2.0 and 2.1. I'll begin with §2.1 on Friday and it will be easier going for you if you've read §2.0 and begun §2.1. (Chapter 1 of Stroock is a review of Riemann integration (§1.1) and a discussion of Riemann-Stieltjes integration (§1.2). The latter material has some importance, but we won't need it for our discussion of Lebesgue integration and you need not worry about it now.)