## Mathematics 105 — Spring 2004 — M. Christ[1]

Solutions to selecta from problem set #2, with problem set #3

**2-5**. Let $f(x,y) = x|y|/\sqrt{x^2+y^2}$ for $(x,y) \neq (0,0)$, and let $f(0,0) = 0$. Show that $f$ is not differentiable at 0. (Note: It is customary to write 0 as shorthand for $(0,0,\cdots,0) \in \mathbb{R}^n$, just as the additive identity element of a vector space is customarily denoted by 0.)

**Solution.** Suppose that $f$ were differentiable at 0. We have $f(0+t,0) - f(0,0) \equiv 0$. Taking $v = (t,0) = te_1$ in the definition of $Df(0)$ would give us

$$0 = \lim_{|t|\to 0} \frac{|f(0+t,0) - f(0,0) - T(te_1)|}{|te_1|} = \lim_{t\to 0} \frac{|T(e_1)|}{1} = |T(e_1)|,$$

so $T(e_1) = 0$. Likewise $f(0,0+t) = f(0,0) \equiv 0$, so the same reasoning gives $T(e_2) = 0$. Since $e_1, e_2$ together form a basis for $\mathbb{R}^2$, this implies that $T = 0$, that is, $T(v) = 0$ for every $v \in \mathbb{R}^2$.

Now consider $v = (t,t)$. We are still assuming that $Df(0)$ exists, and we have proved that if it does, it must equal 0. Therefore

$$0 = \lim_{|t|\to 0} \frac{|f(t,t) - f(0,0) - T(t,t)|}{|t|\sqrt{2}} = \lim_{|t|\to 0} 2^{-1/2} \frac{|f(t,t)|}{|t|} = \lim_{|t|\to 0} 2^{-1/2} \frac{t}{|t|},$$

which is a contradiction. Therefore $Df(0)$ cannot exist. □

**2-7**. Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy $|f(x)| \le |x|^2$. Show that $f$ is differentiable at 0.

**Solution.** This is one of those cases where it's easier to prove more than is asked for: We'll show that $Df(0) = 0$. (As is so often the case in vector calculus, the notation in this equation is imperfect; the zero on the left-hand side of the equation is the origin in $\mathbb{R}^2$, while the zero on the right-hand side refers to the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}$ which satisfies $T(v) = 0$ for all $v \in \mathbb{R}^2$.)

Letting $T$ be the linear transformation 0, for any $x \neq 0$ we have $\frac{|f(x)-f(0)-T(x)|}{|x-0|} = \frac{|f(x)|}{|x|} \le |x|$. This certainly tends to 0 as $x \to 0$, so we've shown that $Df(0) = 0$. □

**2-15(a)**. Define a function $\det : (\mathbb{R}^n)^n \to \mathbb{R}$ by regarding an element $x = (x_1, \cdots, x_n) \in (\mathbb{R}^n)^n$ as a matrix whose rows are the vectors $x_j$, and taking its determinant. Show that det is a differentiable function, and show that

$$D\det(a)(x) = \sum_{i=1}^{n} \det\big(a_1, a_2, \cdots, a_{i-1}, x_i, a_{i+1}, \cdots, a_n\big).$$

(Here $\big(a_1, a_2, \cdots, a_{i-1}, x_i, a_{i+1}, \cdots\big) \in (\mathbb{R}^n)^n$, so we can form its determinant. Note also that the notation (which is commonly used) is slightly misleading when $i = 1$ or $n$.)

*Comment.* In this problem we systematically identify $(\mathbb{R}^n)^n$ with $\mathbb{R}^{n^2}$ via the correspondence $((x_1^1, \cdots, x_1^n), \cdots, (x_n^1, \cdots, x_n^n)) \leftrightarrow (x_1^1, \cdots, x_1^n, x_2^1, \cdots, x_2^n, x_3^1, \cdots)$. And we also identify elements of $(\mathbb{R}^n)^n$ with $n \times n$ matrices, letting $(x_1, \cdots, x_n) \in (\mathbb{R}^n)^n$ correspond with the matrix whose $k$-th row is $x_k = (x_k^1, \cdots, x_k^n)$. Thus det can be regarded either as a function with domain equal to $\mathbb{R}^{n^2}$, or to $(\mathbb{R}^n)^n$, or to the set of all $n \times n$ matrices; we use the same notation for all three.

**Solution.** (Having to type this is my penance for assigning this problem.) Consider $\det(a+x)$ where $a = (a_1, \cdots, a_n)$ and $a_i = (a_i^1, \cdots, a_i^n)$, with corresponding notation for $x$. Recall that $\det(a)$ is a certain polynomial function of the quantities $a_i^j$. In particular, it is a differentiable function; the linear transformation $D\det(a)$ exists for every $a$.

By matrix operations it is clear that

$$\det(a+x) = \det(a) + \sum_{i,j=1}^{n} (-1)^{i+j} x_i^j \det(M_i^j(a)) + P(a,x).$$

where $P(a,x)$ is a polynomial which may be expressed as a sum of monomials, each of which contains <u>at least two factors</u> which are components $x_k^m$ of $x$, and where $M_i^j(a)$ is the $(n-1)\times(n-1)$ minor matrix obtained by deleting from $a$ (regarded as a matrix) the $i$-th row and $j$-th column. To see this just repeatedly apply the rules for expanding determinants in terms of minors.

Therefore if we define $T(x) = \sum_{i,j=1}^{n} (-1)^{i+j} x_i^j \det(M_i^j(a))$ then $T : (\mathbb{R}^n)^n \to \mathbb{R}$ is a linear transformation, and weve shown that $\det(a+x) - \det(a) - T(x) = P(a,x)$. Clearly $|P(a,x)|/|x| \to 0$ as $|x| \to 0$, since $P(a,x)$ is a finite sum of terms of second and higher degrees in the components of $x$. By definition, this means that

$$D\det(a)(x) = \sum_{i,j=1}^{n} (-1)^{i+j} x_i^j \det(M_i^j(a)) \tag{1}$$

If we fix any index $i$ then

$$\sum_{j=1}^{n} (-1)^{i+j} x_i^j \det(M_i^j(a)) = \det\big(a_1, a_2, \cdots, a_{i-1}, x_i, a_{i+1}, \cdots, a_n\big)$$

as one sees by starting on the right-hand side and expanding the determinant using the $i$-th row and the corresponding minors. This together with (1) gives the desired formula. $\square$

**2-15(b)**. With the notation of part (a), suppose that $a_i^j : \mathbb{R}^1 \to \mathbb{R}^1$ are differentiable functions. Let $f(t) = \det(a(t))$. Show that $f'(t) = \sum_{i=}^{n} \det(a_1(t), a_2(t), \cdots, a_{i-1}(t), a_i'(t), a_{i+1}(t), \cdots)$.

**Solution.** $f$ is the composition $\det \circ a : \mathbb{R}^1 \to \mathbb{R}^1$. By the chain rule, $f$ is differentiable, and $f'(t) = \det'(a(t)) \cdot a'(t)$, where the dot denotes multiplication of matrices. Now it's just a matter of getting the notation sufficiently straight to work out this product; it's confusing because we're already secretly thinking of vectors (elements of $\mathbb{R}^{n^2}$) as matrices. I've chosen to consistently write these vectors as vectors, not as matrices.

By Theorem 2-3 part (3), $a'(t)$ is the *transpose* of $(a_1'(t), \cdots, a_n'(t)) \in (\mathbb{R}^n)^n$. Here $a_i'(t) = ((a_i^1)'(t), \cdots, (a_i^n)'(t))$ so $a'(t)$ is the transpose of a row vector which has $n^2$ components; $a'$ is a column vector the domain of $a$ is $\mathbb{R}^1$. On the other hand, det is a function from $(\mathbb{R}^n)^n$ to $\mathbb{R}^1$, so for any $b \in (\mathbb{R}^n)^n$, the Jacobian matrix $\det'(b)$ is a row vector. Thus the product $\det'(a(t)) \cdot a'(t)$ is the product of a row vector (left-hand factor) with a column vector (right-hand factor); their product is a scalar. By (1) and the chain rule, it equals

$\sum_{i,j=1}^n (-1)^{i+j}(a_i^j)'(t)\det(M_i^j(a))$. As we did at the end of part (a), we may rewrite this in the desired form by summing over $j$ for each $i$. □

**2-15(c)**. Let $b_1, \cdots, b_n$ be differentiable functions from $\mathbb{R}^1$ to $\mathbb{R}^1$. Suppose likewise that $a_i : \mathbb{R}^1 \to \mathbb{R}^n$ are differentiable, that $\det(a(t)) \neq 0$ for all $t \in \mathbb{R}$ (or merely for all $t$ in some open interval), and that $s_1(t), \cdots, s_n(t)$ satisfy $\sum_{j=1}^n a_i^j(t)s_j(t) = b_i(t)$ for all $i = 1, 2, \cdots, n$. Show that each $s_i$ is differentiable, and find $s_i'(t)$.

**Solution.** Let $A(t)$ be the $n \times n$ matrix whose $i$-th row is $a_i(t)$; I write capital $A$ to distinguish this matrix from the corresponding element $a$ of $(\mathbb{R}^n)^n$. Since $\det A(t) \neq 0$, $A(t)$ is invertible. Denote by $A^{-1} = A(t)^{-1}$ the inverse matrix. Recall that there is a formula for the inverse of a matrix; each entry of $A(t)^{-1}$ equals the determinant of a certain minor of $A(t)$, divided by $\det A(t)$. Thus each entry of $A(t)^{-1}$ takes the form $P(a_1^1(t), a_1^2(t), \cdots, a_n^n(t))/\det A(t)$ where $P$ is some polynomial. This is a real-valued function of one real variable, so we may apply one-variable calculus to deduce that it is differentiable (use the chain rule, the differentiability of polynomials, and the differentiability of ratios of differentiable functions wherever the denominator does not vanish). Thus each entry of $A^{-1}$ is differentiable.

If we denote by $c_{i,j}(t)$ the entries of $A(t)^{-1}$, then $s_i(t) = \sum_{j=1}^n c_{i,j}(t)b_j(t)$, so $s_i(t)$ is likewise differentiable. Since each component of $s$ is a differentiable function of $t$, so is $s$ (Theorem 2-3 (3)).

Now for the fun part. Here's how to derive a clean formula for the entries $c_{i,j}(t)$ of $A(t)^{-1}$: Consider the equation $\sum_{j=1}^n a_i^j(t)s_j(t) = b_i(t)$, in which all functions map $\mathbb{R}^1$ to $\mathbb{R}^1$ and hence fall within the scope of one variable calculus, and differentiate both sides to obtain

$$\sum_{j=1}^n (a_i^j)'(t)s_j(t) + \sum_{j=1}^n a_i^j(t)s_j'(t) = b_i'(t).$$

If we let $A' = A'(t)$ denote the matrix whose entries are $(a_i^j)'(t)$ then this may be rewritten as the matrix equation $A's + As' = b'$. Substituting $s = A^{-1}b$, this becomes $As' = b' - A'A^{-1}b$, and multiplying through on the left by $A^{-1}$ gives

$$\boxed{s' = A^{-1}b' - A^{-1}A'A^{-1}b.} \tag{2}$$

Here $s' = s'(t)$, $A^{-1} = A(t)^{-1}$, and so on. □

*Comments.* (i) If you try to write this out as a sum over various indices you get a mess! This way one sees that the answer has definite structure. (ii) In the case $n = 1$ we're just dealing with the scalar equation $as = b$. Then $s = b/a$, and the derivative equals $b'a^{-1} - ba'a^{-2}$ by the product rule; this is indeed (2) for $n = 1$.

**2-18(c)**. Find the partial derivatives of $f(x, y) = \int_a^{xy} g$, where $g : \mathbb{R} \to \mathbb{R}$ is some continuous function.

**Solution.** To calculate $D_1 f$ let $y \in \mathbb{R}$ be arbitrary, set $h(x) = f(x, y)$, and recall that $D_1 f(x, y) \equiv h'(x)$. By the fundamental theorem of calculus and the chain rule, $h'(x) = g(xy)\frac{d}{dx}(xy) = yg(xy)$. Thus $D_1 f(x, y) = yg(xy)$. Similarly $D_2 f(x, y) = xg(xy)$. □

**2-22**. Show that if $f : \mathbb{R}^2 \to \mathbb{R}^1$ satisfies $D_2 f(x, y) = 0$ for all $(x, y)$, then $f$ is independent of the second variable. If $D_1 f \equiv D_2 f \equiv 0$, show that $f$ is constant. (Notation: $g \equiv h$ means $g(x) = h(x)$ for all $x$ in whatever domain we are discussing.)

**Solution.** Suppose $D_2 f \equiv 0$. Let $x \in \mathbb{R}^1$ be arbitrary, and define $g(y) = f(x, y)$. Then $g$ is a differentiable function, and $g'(y) = D_2 f(x, y)$; this follows directly from the definition of the directional derivative since $g(y + h) - g(y) = f(x, y + h) - f(x, y)$. Thus $g' \equiv 0$. Any function $g$ from an open one-dimensional interval to $\mathbb{R}$, whose derivative exists at every point and is identically zero, is a constant function (this is proved in Math 1 as a consequence of the mean value theorem). Thus $g(y) = g(0)$, that is, $f(x, y) = f(x, 0)$ for all $y \in \mathbb{R}$. Another way to say this is that $f(x, y_1) = f(x, y_2)$ for all $y_1, y_2 \in \mathbb{R}$; thus $f$ is independent of the second variable.

Now suppose that we are also given that $D_1 f \equiv 0$. We already know that $f(x, y) \equiv f(x, 0)$. Defining $F(x) = f(x, 0)$, we know that $F'(x) \equiv D_1 f(x, 0)$. Thus $F' \equiv 0$, so $F$ is constant, so $f(x, y) = f(x, 0) = F(0)$ for all $(x, y) \in \mathbb{R}^2$. Thus $f$ is constant. □

**2-24(a)**. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0, 0) = 0$, and otherwise $f(x, y) = xy(x^2 - y^2)/(x^2 + y^2)$. Show that $D_2 f(x, 0) \equiv x$ and $D_1 f(0, y) \equiv -y$ for all $x, y$.

**Solution.** I'll just analyze $D_2 f(x, 0)$; the reasoning for the other part is identical (or just notice that $f(x, y) \equiv -f(y, x)$ so that the second conclusion follows from the first). First consider the case where $x \neq 0$. In that case we can calculate $D_2 f(x, 0)$ by setting $g(y) = f(x, y)$, and evaluating $g'(0)$ using one-variable calculus. This is very easy; $g'(y)$ is messy but if we view $g$ as the product of $y$ with $x(x^2 - y^2)/(x^2 + y^2)$ and apply Leibniz's rule then there are two terms; the complicated one where the derivative falls on the second factor can be disregarded since it is multiplied by $y$ and we are going to evaluate it at $y = 0$. In the other term the derivative falls on $y$, and when we set $y = 0$ in the second factor we get $g'(0) = x$ as desired.

To evaluate $D_2 f(0, 0)$ we resort to the definition; for $h \neq 0$ we have $\frac{f(0,h) - f(0,0)}{h} = 0$ since $f(0, h) \equiv 0 = f(0, 0)$ by the definition of $f$. Taking the limit as $h \to 0$ we conclude that $D_2 f(0, 0) = 0$. □

**Problem set 3**: For Friday 2/13, please solve the following problems from Chapter 2 of Spivak: 28(b),29,35,36,37(a),40,41.