Mathematics 105 — Spring 2004 — M. Christ

A Supplementary Note, Selected Solutions for Problem Set 10, and Problem Set 11

Proposition. Denote Lebesgue measure in $\mathbb{R}^k$ by λ_k, and as usual let $\mathcal{B}_{\mathbb{R}^k}, \overline{\mathcal{B}}_{\mathbb{R}^k}$ denote the classes of Borel and Lebesgue measurable sets, respectively. For any $m, n \geq 1$,

$$\mathcal{B}_{\mathbb{R}^m} \times \mathcal{B}_{\mathbb{R}^n} = \mathcal{B}_{\mathbb{R}^{m+n}}. \tag{1}$$

For any set $A \in \mathcal{B}_{\mathbb{R}^{m+n}}$, $\lambda_{m+n}(A) = (\lambda_m \times \lambda_n)(A)$. Finally

$$\overline{\overline{\mathcal{B}}_{\mathbb{R}^m} \times \overline{\mathcal{B}}_{\mathbb{R}^n}} = \overline{\mathcal{B}}_{\mathbb{R}^{m+n}}. \tag{2}$$

□

This proposition allows us to exploit Fubini's and Tonelli's theorems in conjunction with the product structure of $\mathbb{R}^{m+n}$; without it we wouldn't know that the product measure and σ-algebra which arise in Fubini's theorem have any relevance to the "real world" of Lebesgue integration on Euclidean space.

Proof. I'll simplify notation by systematically writing $\mathcal{B}_k$ as an abbreviation for $\mathcal{B}_{\mathbb{R}^k}$. $\mathcal{B}_m \times \mathcal{B}_n$ contains all products $A_1 \times A_2$ with $A_1 \in \mathcal{B}_m, A_2 \in \mathcal{B}_n$. In particular, it contains all such products in which both factors are open. Since any open set in $\mathbb{R}^{m+n}$ can be expressed as a countable union of open squares, $\mathcal{B}_m \times \mathcal{B}_n$ contains all open subsets of the product space; since it's a σ-algebra, it therefore contains all Borel subsets. Thus $\mathcal{B}_m \times \mathcal{B}_n \supset \mathcal{B}_{m+n}$.

To show the reverse inclusion, it suffices to show that $\mathcal{B}_{m+n}$ contains all products $A_1 \times A_2$ with $A_1 \in \mathcal{B}_m, A_2 \in \mathcal{B}_n$, since $\mathcal{B}_m \times \mathcal{B}_n$ is by its definition contained in any σ-algebra which includes all these sets. All that is immediately obvious is that $\mathcal{B}_{m+n}$ contains all products for which both sets A_j are *open*.

Fix an open set A_2 and consider the collection $\mathcal{C}$ of all sets $B \subset \mathbb{R}^m$ such that $B \times A_2 \in \mathcal{B}_{m+n}$. $\mathcal{C}$ contains all open sets B. $\mathcal{C}$ is a σ-algebra: it contains the open sets $\emptyset$ and $\mathbb{R}^m$; it's closed under countable unions since $(\cup_j B_j) \times A_2 = \cup_j (B_j \times A_2)$. To show that it's closed under complementation let $S \in \mathcal{C}$. Note that

$$\mathbb{R}^{m+n} \setminus (S \times A_2) = \big[(\mathbb{R}^m \setminus S) \times A_2\big] \cup \big[\mathbb{R}^m \times (\mathbb{R}^n \setminus A_2)\big], \tag{3}$$

and this is a *disjoint* union. The left-hand side is the complement of a set in $\mathcal{B}_{m+n}$, so belongs to $\mathcal{B}_{m+n}$. The second set on the right is the Cartesian product of two closed sets, so is closed, so belongs to $\mathcal{B}_{m+n}$. Therefore $(\mathbb{R}^m \setminus S) \times A_2 \in \mathcal{B}_{m+n}$, as desired.

Since $\mathcal{C}$ is a σ-algebra which contains all open sets, it contains all of $\mathcal{B}_m$. Thus we conlude that if $A_1 \in \mathcal{B}_m$ and if $A_2 \subset \mathbb{R}^n$ is open, then $A_1 \times A_2 \in \mathcal{B}_{m+n}$. Of course, the same conclusion then holds with the roles of $\mathbb{R}^m, \mathbb{R}^n$ reversed.

Now fix any $A_2 \in \mathcal{B}_n$ and redefine $\mathcal{C}$ to be the collection of all $A_1 \subset \mathbb{R}^m$ such that $A_1 \times A_2 \in \mathcal{B}_{m+n}$. By repeating the above reasoning we find that $\mathcal{C}$ is a σ-algebra; the only modification is that the set $\mathbb{R}^m \times (\mathbb{R}^n \setminus A_2)$ is no longer closed. However, it is the product of an open set with a Borel set, so by the preceding step belongs to $\mathcal{B}_{m+n}$. Thus $\mathcal{C}$ is a σ-algebra, and since it contains all open sets (again by the preceding step), it contains all Borel sets. Thus we've shown that the product of any two Borel sets belongs to the σ-algebra $\mathcal{B}_{m+n}$. Now $\mathcal{B}_m \times \mathcal{B}_n$ is the smallest σ-algebra which contains all products of Borel sets, so it must be contained in $\mathcal{B}_{m+n}$. This completes the proof that $\mathcal{B}_m \times \mathcal{B}_n = \mathcal{B}_{m+n}$. □

Now we have two measures, $\mu = \lambda_{m+n}$ and $\nu = \lambda_m \times \lambda_n$, on the same σ-algebra $\mathcal{B}_{m+n}$. Define $\mathcal{A}$ to be the collection of all sets $A \in \mathcal{B}_{m+n}$ satisfying $\lambda_{m+n}(A) = (\lambda_m \times \lambda_n)(A)$. Define $\mathcal{C}$ to be the collection of all products $A_1 \times A_2$ for which both factors A_j are open. Then $\mathcal{C}$ is a π-system, and our two measures agree on $\mathcal{C}$, that is, $\mathcal{C} \subset \mathcal{A}$.

I claim that $\mathcal{A}$ is a λ-system.[1] (i) $\mathbb{R}^{m+n} \in \mathcal{C} \subset \mathcal{A}$ since $\mathbb{R}^{m+n}$ is a product of open sets. (ii) If $A, B \in \mathcal{A}$ are disjoint then

$$\lambda_{m+n}(A \cup B) = \lambda_{m+n}(A) + \lambda_{m+n}(B) = \lambda_m \times \lambda_n(A) + \lambda_m \times \lambda_n(B) = \lambda_m \times \lambda_n(A \cup B).$$

(iiia) If $A \subset B$ both belong to $\mathcal{A}$, and if A has finite measure, then $\lambda_{m+n}(B \setminus A) = \lambda_m \times \lambda_n(B \setminus A)$ by an argument almost identical to that just used for disjoint unions; see class notes. (iiib) If $A \subset B$ but A has infinite measure then this argument breaks down (we can't subtract $\lambda_{m+n}(A)$ from both sides of an equation with impunity if it's infinite). I'll return to this point momentarily. (iv) If $A_1 \subset A_2 \subset A_3 \subset \cdots$ is an infinite sequence of nested sets in $\mathcal{A}$, and if $A = \cup_j A_j$ is their ascending limit, then $\lambda_{m+n}(A) = \lim_{j\to\infty} \lambda_{m+n}(A_j) = \lim_{j\to\infty} \lambda_m \times \lambda_n(A_j) = \lambda_m \times \lambda_n(A)$, so $A \in \mathcal{A}$ as well. Thus $\mathcal{A}$ is indeed a λ-system.

Thus (once the difficulty in (iiib) has been resolved) the smallest λ-system containing $\mathcal{C}$ is contained in $\mathcal{A}$. But by Lemma 3.1.3, this smallest λ-system is precisely the smallest σ-algebra containing the π-system $\mathcal{C}$, and we've seen that this σ-algebra is $\mathcal{B}_{m+n}$. Therefore our two measures agree on $\mathcal{B}_{m+n}$. □

As you've no doubt realized, point (iiib) wouldn't be an obstacle of we were working with *finite* measures; moreover we are working with σ-finite measures, the next best thing. To complete the proof we could fix any two bounded open sets $G_m \subset \mathbb{R}^m$, $G_n \subset \mathbb{R}^n$, and modify our measures by defining $\tilde{\lambda}_{m+n}(A) = \lambda_{m+n}(A \cap (G_m \times G_n))$, and likewise for $\widetilde{\lambda_m \times \lambda_n}$. Both of these are *finite* measures on $\mathcal{B}_{m+n}$, so point (iiib) is no longer relevant and the above reasoning proves that they agree on $\mathcal{B}_{m+n}$. By expressing $\mathbb{R}^{m+n}$ as an ascending limit of products $G_{m,j} \times G_{n,j}$ and passing to the limit, we conclude that $\lambda_{m+n}, \lambda_m \times \lambda_n$ agree on all Borel sets. □

The last conclusion to be verified is the identity relating Lebesgue sets. The completion of a σ-algebra is defined in terms of a measure; if we complete the same σ-algebra with respect to different measures, then we could obtain different completions.

[1] cf. Exercise 3.1.8 in our text.

But this point is no danger in the present context since we've already shown that the two measures in question agree. Thus since $\mathcal{B}_{m+n} = \mathcal{B}_m \times \mathcal{B}_n$, we may conclude that $\overline{\mathcal{B}}_{m+n} = \overline{\mathcal{B}_m \times \mathcal{B}_n}$, the completion being taken with respect to $\lambda_{m+n} = \lambda_m \times \lambda_n$. Clearly $\overline{\mathcal{B}_m \times \mathcal{B}_n} \subset \overline{\overline{\mathcal{B}}_m \times \overline{\mathcal{B}}_n}$, so $\overline{\mathcal{B}}_{m+n} \subset \overline{\overline{\mathcal{B}}_m \times \overline{\mathcal{B}}_n}$.

For the reverse inclusion it suffices to show that $\overline{\mathcal{B}}_m \times \overline{\mathcal{B}}_n \subset \overline{\mathcal{B}}_{m+n}$, for the right-hand side is complete and hence[2] the completion of the left-hand side would then be forced to lie in the right-hand side, as well. Moreover, it suffices to show that whenever $A_1 \in \overline{\mathcal{B}}_m$ and $A_2 \in \overline{\mathcal{B}}_n$, $A_1 \times A_2 \in \overline{\mathcal{B}}_{m+n}$, for $\overline{\mathcal{B}}_m \times \overline{\mathcal{B}}_n$ is by definition the smallest σ-algebra containing all such products.

For any such sets A_1, A_2, there exist "sandwiching" Borel sets satisfying $F_j \subset A_j \subset G_j$ and $\lambda(G_j \setminus F_j) = 0$ where $\lambda = \lambda_m$ or $= \lambda_n$, as appropriate. Thus $F_1 \times F_2 \subset A_1 \times A_2 \subset G_1 \times G_2$. $F_1 \times F_2$ and $G_1 \times G_2$ belong to $\mathcal{B}_m \times \mathcal{B}_n$, hence to $\mathcal{B}_{m+n}$. Moreover

$$\lambda_{m+n}(G_1 \times G_2) = \lambda_m(G_1) \cdot \lambda_n(G_2) = \lambda_m(F_1) \cdot \lambda_n(F_2) = \lambda_{m+n}(F_1 \times F_2).$$

Thus $A_1 \times A_2$ is itself sandwiched between Borel sets having equal measures, so $A_1 \times A_2 \in \overline{\mathcal{B}}_{m+n}$, as was to be shown. □

Solutions for Problem Set 10

X.A Here are two associative laws: Let $(E_j, \mathcal{A}_j, \mu_j)$, for $j = 1, 2, 3$, be three σ-finite measure spaces. Prove that the two σ-algebras $(\mathcal{A}_1 \times \mathcal{A}_2) \times \mathcal{A}_3$ and $\mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3)$ (both of which are σ-algebras of subsets of $E_1 \times E_2 \times E_3$) are equal. Prove that the two measures $(\mu_1 \times \mu_2) \times \mu_3$ and $\mu_1 \times (\mu_2 \times \mu_3)$ on $E_1 \times E_2 \times E_3$ are equal.

Solution. This one was fun. To orient ourselves, let's fix firmly in mind the principle that we know *nothing* about a product σ-algebra $\mathcal{A}_1 \times \mathcal{A}_2$, except that it's the smallest σ-algebra which contains certain sets; all our reasoning must be based on this single fact. Let's also recall that $\mathcal{A}_1 \times \mathcal{A}_2$ contains (in general) much more general sets than merely products $A_1 \times A_2$.

I want to show that[3] whenever $S \in \mathcal{A}_1 \times \mathcal{A}_2$ and $A_3 \in \mathcal{A}_3$, necessarily $S \times A_3 \in \mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3)$. Once this has been proved for arbitrary $A_3 \in \mathcal{A}_3$, we can conclude that $(\mathcal{A}_1 \times \mathcal{A}_2) \times \mathcal{A}_3 \subset \mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3)$, for the right-hand side is a σ-algebra, while the left-hand side is the *smallest* σ-algebra containing all products $S \times A_3$ with $S \in \mathcal{A}_1 \times \mathcal{A}_2$ and $A_3 \in \mathcal{A}_3$. Then the reverse inclusion $\mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3) \subset (\mathcal{A}_1 \times \mathcal{A}_2) \times \mathcal{A}_3$ would follow by symmetry, and the proof of equality of our two σ-algebras would be complete.

Fix any set $A_3 \in \mathcal{A}_3$. Define $\mathcal{C}$ to be the collection of all sets $S \subset E_1 \times E_2$ such that $S \times A_3 \in \mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3)$. Certainly $\mathcal{C}$ contains all three-fold products $A_1 \times A_2$ with $A_j \in \mathcal{A}_j$, for $(A_1 \times A_2) \times A_3 = A_1 \times (A_2 \times A_3)$, and $A_2 \times A_3 \in \mathcal{A}_2 \times \mathcal{A}_3$

[2]If $(E, \overline{\mathcal{A}}, \overline{\mu})$ is the completion of a measure space $(E, \mathcal{A}, \mu)$, then $\overline{\mathcal{A}}$ is the σ-algebra generated by $\mathcal{A}$ together with all subsets of null sets in $\mathcal{A}$, that is, the smallest σ-algebra which contains all elements of $\mathcal{A}$ and all subsets of null sets in $\mathcal{A}$.

[3]What's obvious here is that $S \times A_3$ belongs to the other σ-algebra, $(\mathcal{A}_1 \times \mathcal{A}_2) \times \mathcal{A}_3$.

by definition of the product σ-algebra. $\mathcal{C}$ is itself a σ-algebra. For if $S_j \in \mathcal{C}$ then $(\cup_j S_j \times A_3) = \cup_j (S_j \times A_3)$, which is a countable union of elements of $\mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3)$, hence is an element of $\mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3)$. And if $S \in \mathcal{C}$ then $(E_1 \times E_2) \setminus S \in \mathcal{C}$ by an identity analogous to (3).

Since $\mathcal{C}$ is a σ-algebra containing all products $A_1 \times A_2$ with $A_j \in \mathcal{A}_j$, it contains the smallest such σ-algebra, which is $\mathcal{A}_1 \times \mathcal{A}_2$. Thus the proof is complete. $\square$

Define $\mathcal{A} = \mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3) = \mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3)$. Before going on, let's observe that only two properties of $\mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3)$ were actually used in this argument: (i) it contains all triple products $A_1 \times A_2 \times A_3$ such that each $A_j \in \mathcal{A}_j$, and (ii) it is a σ-algebra. Thus we've proved that *any* σ-algebra with those two properties contains $\mathcal{A}$. Therefore $\mathcal{A}$ equals the smallest σ-algebra that contains all such triple products. $\square$

The next step is to show that the two measures $\lambda = (\mu_1 \times \mu_2) \times \mu_3$ and $\nu = \mu_1 \times (\mu_2) \times \mu_3)$ agree on $\mathcal{A} = (\mathcal{A}_1 \times \mathcal{A}_2) \times \mathcal{A}_3 = \mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3)$. Let $\mathcal{C}$ be the collection of all triple products $A_1 \times A_2 \times A_3$ with each $A_j \in \mathcal{A}_j$. Then $\mathcal{C}$ is a π-system, and $\lambda(A) = \mu(A)$ for all $A \in \mathcal{C}$. Consider the collection $\mathcal{B}$ of all sets $A \in \mathcal{A}$ for which $\lambda(A) = \nu(A)$. Assuming for a moment that each measure μ_j is finite, it follows from reasoning employed in the proof of the Proposition above that $\mathcal{B}$ is a λ-system. Thus λ, ν agree on some λ-system containing $\mathcal{C}$; therefore they agree on the smallest such λ-system; but by Lemma 3.1.3, that smallest λ-system is the smallest σ-algebra containing $\mathcal{C}$. We've already proved that this smallest σ-algebra is $\mathcal{A}$.

The case of σ-finite measure spaces follows from the case of finite spaces, by a simple argument which is left to the reader. $\square$

X.B Another example: Let $(E_j, \mathcal{A}_j, \mu_j)$ equal $\mathbb{N}$ equipped with the σ-algebra of all its subsets, and with counting measure. (Thus $\mathcal{A}_1 \times \mathcal{A}_2$ consists of all subsets of $\mathbb{N} \times \mathbb{N}$.) Define $f(x_1, x_2)$ to be 1 if $x_1 = x_2 \in \mathbb{N}$, to be -1 if $x_2 = x_1 + 1$, and to be 0 otherwise. Show that $\iint f \, d\mu_1 \, d\mu_2 \neq \iint f \, d\mu_2 \, d\mu_1$. Which hypothesis of Fubini's theorem is not satisfied?

Solution.

$$\int f(x_1, x_2) \, d\mu_1(x_1) = \sum_{x_1 \in \mathbb{N}} f(x_1, x_2) = \begin{cases} 1 & \text{if } x_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\int \int f(x_1, x_2) \, d\mu_1(x_1) \, d\mu_2(x_2) = \sum_{x_2 \in \mathbb{N}} \sum_{x_1 \in \mathbb{N}} f(x_1, x_2) = 1.$$

Since $\sum_{x_2 \in \mathbb{N}} f(x_1, x_2) = 0$ for every $x_1 \in \mathbb{N}$, integrating in the reverse order gives $0 \neq 1$. $\square$

The function f does not belong to $L^1(\mathbb{N} \times \mathbb{N})$, so Fubini's theorem does not apply; likewise it is not nonnegative, so Tonelli's theorem doesn't apply, either. $\square$

X.C As a complement to problem 4.1.10, show that if $(E_j, \mathcal{A}_j, \mu_j) = (\mathbb{R}^1, \overline{\mathcal{B}}_{\mathbb{R}^1}, \lambda)$ where λ denotes Lebesgue measure on $\mathbb{R}^1$, then the product measure $\mu_1 \times \mu_2$ on $\mathcal{A}_1 \times \mathcal{A}_2$ is *not* complete.

Solution. I've essentially done this in class. Choose any nonmeasurable set $A \subset \mathbb{R}^1$. The set $\{0\} \times A$ is contained in $\{0\} \times \mathbb{R}^1$, which is a null set. On the other hand, $\{0\} \times A$ cannot belong to the product σ-algebra $\overline{\mathcal{B}}_{\mathbb{R}^1} \times \overline{\mathcal{B}}_{\mathbb{R}^1}$. For if it did, then its intersection with any vertical line would be forced to be a measurable set, that is, to belong to $\overline{\mathcal{B}}_{\mathbb{R}^1}$, by one of the conclusions of Fubini's theorem. But the intersection with the vertical line $\{0\} \times \mathbb{R}^1$ is not measurable. $\square$

Note that problem X.C demonstrates that $\overline{\mathcal{B}}_{\mathbb{R}^1} \times \overline{\mathcal{B}}_{\mathbb{R}^1} \neq \overline{\overline{\mathcal{B}}_{\mathbb{R}^1} \times \overline{\mathcal{B}}_{\mathbb{R}^1}}$.

X.D Earlier in the course we sweated quite a bit to prove that $|T(A)| = |\det(T)| \cdot |A|$ for any measurable set $A \subset \mathbb{R}^n$ and any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. One part of this argument can now be reworked as a consequence of Fubini's Theorem (and problem 4.1.10), as follows. (For simplicity I'll restrict the problem to Borel sets; the extension to Lebesgue measurable sets follows fairly easily from the fact that any such set is the union of a Borel set with a set of measure zero. Likewise I'll simplify the discussion by assuming T to be invertible.)

We take for granted the fact that any linear transformation of $\mathbb{R}^n$ can be expressed as a finite composition of linear transformations of three basic types[4]: (i) Dilations $x = (x_1, \cdots, x_n) \mapsto (cx_1, x_2, \cdots, x_n)$ for some $c \neq 0$; (ii) Permutations $x \mapsto (x_k, \cdots, x_{k-1}, x_1, x_{k+1}, \cdots, x_n)$; and (iii) Shear transformations $x \mapsto (x_1, \cdots, x_{k-1}, x_k + cx_1, x_{k+1}, \cdots, x_n)$. (In (ii) and (iii), $k \in \{1, 2, \cdots, n\}$ is arbitrary.) Let's take for granted the result for linear transformations of the first two types, and let's simplify the notation by discussing only dimension $n = 2$.

Show that for any shear transformation T of $\mathbb{R}^2$, $|T(A)| = |A|$ for all Borel subsets A of $\mathbb{R}^2$. (Use Fubini's theorem, not Theorem 2.2.2!) Explain briefly why this leads to the conclusion $|T(A)| = |\det(T)| \cdot |A|$ (for all Borel subsets A of $\mathbb{R}^2$), for all invertible linear transformations T of $\mathbb{R}^2$.

Solution. I'll use the following **Lemma**: For any $f \in L^1(\mathbb{R}^n)$, and for any $z \in \mathbb{R}^n$, the function $g(x) = f(x - z)$ belongs to L^1 and satisfies $\int_{\mathbb{R}^n} g\, d\lambda = \int_{\mathbb{R}^n} f\, d\lambda$, where λ denotes Lebesgue measure. If f is the characteristic function of a measurable set, this simply amounts to the translation-invariance of Lebesgue measure, which we've already proved. The identity then follows for simple functions, then nonnegative functions, then L^1 functions, by what are now very familiar arguments. $\square$

Let $A \subset \mathbb{R}^2$ be any Borel set, and let $c \in \mathbb{R}$. Define $T(x_1, x_2) = (x_1, x_2 + cx_1)$, and note that $T^{-1}(x_1, x_2) = (x_1, x_2 - cx_1)$, and that $\det(T) = 1$. Note also that $T(A) = (T^{-1})^{-1}(A)$ is a Borel set, since A is Borel and T^{-1} is continuous. Note finally that $T(A) = \{x = (x_1, x_2) : T^{-1}(x) \in A\}$ so that $(x_1, x_2) \in T(A)$ if and only if $(x_1, x_2 - cx_1) \in A$. Thus

$$\chi_{T(A)}(x) = \chi_A(x_1, x_2 - cx_1).$$

[4]This just amounts to the fact that any invertible matrix can be reduced to the identity matrix via the usual row and column operations.

Now writing λ_k to denote Lebesgue measure in $\mathbb{R}^k$,

$$\begin{aligned}\lambda(T(A)) &= \int_{\mathbb{R}^2} \chi_{T(A)}(x)\, d\lambda_2(x) \\ &= \int_{\mathbb{R}^2} \chi_A(x_1, x_2 - cx_1)\, d\lambda_2(x) = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \chi_A(x_1, x_2 - cx_1)\, d\lambda_1(x_2) \Big)\, d\lambda_1(x_1)\end{aligned}$$

by Tonelli's theorem. By the above lemma, the inner integral is $\int_{\mathbb{R}} \chi_A(x_1, x_2)\, d\lambda_1(x_2)$, so by applying Tonelli in reverse we obtain

$$\lambda(T(A)) = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \chi_A(x_1, x_2)\, d\lambda_1(x_2) \Big)\, d\lambda_1(x_1) = \lambda(A).$$

This leads to the conclusion that $|T(A)| = |\det(T)| \cdot |A|$ for all invertible linear transformations T, because any such T can be factored as a finite composition as $T = T_1 \circ T_2 \circ \cdots T_N$ where T_j are various linear transformations for which we already know this identity to be valid. Then

$$\begin{aligned}|T(A)| &= |T_1(T_2 \circ \cdots \circ T_N(A))| = |\det(T_1)| \cdot |T_2 \circ \cdots \circ T_N(A)| \\ &= |\det(T_1)| \cdot |\det(T_2)| \cdot |T_3 \circ \cdots \circ T_N(A)| = \cdots = \prod_j |\det(T_j)| \cdot |A|.\end{aligned}$$

Since $\det(T) = \prod_j \det(T_j)$, the result follows. □

X.E We have studied the set $L^1 = L^1(\mathbb{R}^n, \overline{\mathcal{B}}_{\mathbb{R}^n}, \lambda)$ of all Lebesgue measurable, integrable functions from $\mathbb{R}^n$ to the extended real numbers, and have learned that this set is a complete normed linear space when equipped with the L^1 norm. In this problem we will learn that this set of functions actually has the structure of an *algebra*.[5] You might guess that the algebra structure would be defined by pointwise multiplication, the product of f, g being $h(x) = f(x)g(x)$, but it turns out (see part (a) below) that this pointwise product of two L^1 functions need not belong to L^1. Instead, the product is defined formally by

$$f * g(x) = \int f(x-y)g(y)\, d\lambda(y) \tag{4}$$

where λ denotes Lebesgue measure on $\mathbb{R}^n$.

X.E(a) Let f be defined on $\mathbb{R}^1$ by $f(x) = x^{-1/2}$ for all $x \in (0, 1]$ and $= 0$ for all other x. Show that $f \in L^1$, but the function $x \mapsto f(x)^2$ does not belong to L^1.

Solution. Let $\delta \in (0, 1)$. The function $x^{-1/2}$ is continuous on $[\delta, 1]$, so its Lebesgue integral equals its Riemann integral, which is $2x^{1/2}\big|_\delta^1 = 2 - 2\delta^{1/2}$. Letting $\delta = 1/n$ where $n \in \mathbb{N}$, and applying the monotone convergence theorem to $f_n(x) = x^{-1/2}\chi_{[1/n,1]}$, we conclude that the Lebesgue integral of f equals 2.

[5] This algebra structure arises naturally in many applications, for instance in the analysis of the basic partial differential equations of mathematical physics.

The same reasoning shows that the Lebesgue integral of f^2 is infinite, since the Riemann integral of x^{-1} over $[\delta, 1]$ is $\ln(1/\delta)$. $\square$

X.E(b) In the rest of the problem we fix a dimension n and use both x and y for coordinates on $\mathbb{R}^n$. Thus (x, y) denotes a point in $\mathbb{R}^n \times \mathbb{R}^n$, which we identify with $\mathbb{R}^{2n}$ in the natural way. Show that if $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is *Borel* measurable, then the function $F(x, y) = f(x - y)$ is a Borel measurable function on $\mathbb{R}^{2n}$.

Solution. We know that the inverse image of any Borel set under any *continuous* mapping is a Borel set. $F = f \circ \varphi$, so for any $a \in \mathbb{R}$, $F^{-1}((a, +\infty]) = \varphi^{-1}(f^{-1}((a, +\infty]))$. The map $\varphi(x, y) = x - y$ is continuous, so this is the inverse image, under the continuous function φ, of the Borel set $f^{-1}((a, +\infty])$. $\square$

X.E(c) Show that if $A \subset \mathbb{R}^n$ is Lebesgue measurable with measure 0, then $\tilde{A} = \{(x, y) \in \mathbb{R}^{2n} : x - y \in A\}$ has exterior measure zero, and therefore is Lebesgue measurable.

Solution. Let $A \subset \mathbb{R}^n$ be any Lebesgue measurable null set. Consider any large finite radius R and let $S_R = \{(x, y) : |y| < R\}$; it suffices to show that the exterior measure of $\tilde{A} \cap S_R$ is arbitrarily small, for then $\tilde{A} \cap S_R$ is a null set and by taking the union over a sequence of radii R tending to ∞ we could conclude that $\tilde{A}$ is a countable union of null sets, hence is itself a null set.

First let's consider an incorrect solution: Writing λ to denote Lebesgue measure both in $\mathbb{R}^{2n}$ and in $\mathbb{R}^n$,

$$|\tilde{A}| = \int_{\mathbb{R}^{2n}} \chi_A(x - y)\, d\lambda(x, y) = \int \int \chi_A(x - y)\, d\lambda(x)\, d\lambda(y)$$
$$= \int \int \chi_A(x)\, d\lambda(x)\, d\lambda(y) = \int |A|\, d\lambda(y) = \int_{\mathbb{R}^n} 0\, d\lambda(y) \ = 0.$$

I've used Tonelli's theorem, and then the Lemma in problem X.D to evaluate the inner integral. The problem with this proof is that we don't know that $\tilde{A}$ is a Lebesgue measurable set, which we need in order to invoke Tonelli's theorem.

The natural thing to do is to approximate. Let $\varepsilon > 0$, and choose an open set $G \supset A$ satisfying $|G| < \varepsilon$. Define $\tilde{G} = \{(x, y) : x - y \in G\}$. Now $\tilde{G}$ is open, so the above reasoning applies to it. Alas, all we get is that

$$|\tilde{G}| = \int_{\mathbb{R}^n} \varepsilon\, d\lambda(y) = \varepsilon |\mathbb{R}^n| = +\infty,$$

no matter how small ε is.

The purpose of the sets S_R suggested in the hint is to sidestep the factor of $+\infty = |\mathbb{R}^n|$ which arises in the above argument. Let $\varepsilon > 0$ be arbitrary. Choose an open set $G \supset A$ satisfying $|G| < \varepsilon$. Then $\tilde{A} \cap S_R \subset \tilde{G} \cap S_R$. $\tilde{G}$ is a Borel set, so its characteristic function is (Lebesgue) measurable and we may apply Tonelli's theorem to it. Therefore

$$|\tilde{G} \cap S_R| = \int_{\mathbb{R}^{2n}} \chi_{\tilde{G}}(x,y)\chi_{S_R}(x,y)\,d\lambda(x,y)$$
$$= \int \chi_G(x-y)\chi_{S_R}(x,y)\,d\lambda(x,y) = \int_{|y|\le R} \Big(\int \chi_G(x-y)\,d\lambda(x)\Big)\,d\lambda(y).$$

By the lemma discussed at the beginning of the solution for problem X.D, the inner integral is simply $\int \chi_G(x)\,d\lambda(x) = |G| < \varepsilon$. Since this is independent of y we may then factor it out, and the remaining integral is then simply $|S_R|$. Thus we've shown that $|\tilde{A}\cap S_R|_e \le |\tilde{G}\cap S_R| = |G|\cdot|S_R| < \varepsilon|S_R|$. Since $\varepsilon > 0$ was arbitrary, $|\tilde{A}\cap S_R|_e = 0$. □

X.E(d) Deduce from the preceding that if $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is Lebesgue measurable, then the function defined by $F(x,y) = f(x-y)$ is Lebesgue measurable.

Solution. Let $a \in \mathbb{R}$, let $I = (a, +\infty]$, and consider $f^{-1}(I)$. This is by hypothesis a Lebesgue measurable set, so we may express it as a disjoint union $B \cup A$ where B is a Borel set, and A is a null set (with respect to Lebesgue measure in $\mathbb{R}^n$). Now $F^{-1}(I) = \tilde{A} \cup \tilde{B}$, where the notation $\tilde{\cdot}$ has the same same meaning as in part (c). The set $\tilde{B} \subset \mathbb{R}^{2n}$ is Borel measurable, hence Lebesgue measurable, as shown in part (b), while $\tilde{A}$ is a null set, hence a Lebesgue measurable set, by part (c). □

X.E(e) For the remainder of the problem assume that $f, g \in L^1(\mathbb{R}^n)$. Show[6] that the function $h(x,y) = f(x-y)g(y)$ is Lebesgue measurable (on $\mathbb{R}^{2n}$, of course).

Solution. There's little to do here; we've proved that f is measurable, measurability of the function $G(x,y) = g(y)$ is obvious since $G^{-1}(I) = \mathbb{R}^n \times g^{-1}(I)$ and the Cartesian product of any two Lebesgue measurable sets is Lebesgue measurable. Finally the product of any two measurable functions is always measurable. □

X.E(f) Show further that $h \in L^1(\mathbb{R}^{2n})$.

Solution. By Tonelli's theorem,

$$\int_{\mathbb{R}^{2n}} |f(x-y)g(y)|\,d\lambda(x,y) = \int\int |f(x-y)g(y)|\,d\lambda(x)\,d\lambda(y)$$
$$= \int\int |f(t)|\,d\lambda(t)\,|g(y)|\,d\lambda(y) = \int \|f\|_{L^1}|g(y)|\,d\lambda(y) = \|f\|_{L^1}\|g\|_{L^1}.$$

Therefore $h \in L^1(\mathbb{R}^{2n})$. □

X.E(g) Show that for almost every $x \in \mathbb{R}^n$, the function $y \mapsto f(x-y)g(y)$ belongs to $L^1(\mathbb{R}^n)$, and moreover that $x \mapsto \int f(x-y)g(y)\,d\lambda(y)$ defines a measurable function in $L^1(\mathbb{R}^n)$. Thus we've shown that (4) does define an element of $L^1(\mathbb{R}^n)$, for arbitrary $f, g \in L^1(\mathbb{R}^n)$.

Solution. With the conclusion of part (f) in hand, this is a direct consequence of Fubini's theorem. □

[6] In an earlier draft I wrote $|f(x-y)g(y)|$ only because I stupidly forgot that in this course, we've agreed that the product of any two extended real numbers is defined; I was being ultra-cautious by excluding from consideration products of plus infinity with minus infinity.

X.E(h) Show that for any $f, g \in L^1(\mathbb{R}^n)$, $\int f * g \, d\lambda = \int f \, d\lambda \cdot \int g \, d\lambda$, and that $\|f * g\|_{L^1} \leq \|f\|_{L^1} \cdot \|g\|_{L^1}$.

Solution. Another direct application of Fubini; see part (f) of this problem. □

X.E(i) Show that $f*g = g*f$ (in the almost-everywhere sense). (You are permitted to use without proof any reasonable change-of-variables formula that you need to derive this.) Similar reasoning shows that the product $*$ satisfies the associative law, but you're not asked to prove this.

Solution. Given any $x \in \mathbb{R}^n$, make the substitution $y \mapsto z = x - y$; thus $y = x - z$ and

$$f*g(x) = \int f(x-y)g(y) \, d\lambda(y) = \int f(z)g(x-z) \, d\lambda(z) = \int g(x-z)f(z) \, d\lambda(z) = g*f(x).$$

This holds for any x for which the integrand is in $L^1(\mathbb{R}^n)$, hence (by earlier parts of this problem) for almost every $x \in \mathbb{R}^n$.

The third = sign is justified by the commutativity of multiplication of (extended) real numbers. The second = sign is justified by the change of variables

$$\int \varphi(x-y) \, d\lambda(y) = \int \varphi(z) \, d\lambda(z).$$

If a proof of this had been required, we would have shown firstly that it holds for characteristic functions of measurable sets, then for simple functions, then for nonnegative functions, and finally for integrable functions. The initial step, validity for characteristic functions of measurable sets, holds because Lebesgue measure is invariant under translation, and is also invariant under the linear transformation $T(y) = -y$ since T has determinant $= (-1)^n$. □

Comment. Here's a point that can easily lead to confusion. Consider, in $\mathbb{R}^1$, the Lebesgue integral $\int_{[a,b]} f(x) \, d\lambda(x)$. Make the change of variables $x = -y$. Does our integral become $\int_{[-b,-a]} f(-y) \, d\lambda(y)$, or have we lost a minus sign? After all, if f were continuous and if we were doing Riemann integration then we'd have $\int_a^b f(x) \, dx = \int_{-a}^{-b} f(-y) \cdot (-\,dy)$ by the usual change-of-variables formula; we seem to be off by a minus sign.

There are several explanations. On the one hand, if we continued the Riemann integral calculation we'd note that $\int_{-a}^{-b} f(-y) \, dy = -\int_{-b}^{-a} f(-y) \, dy$, thus obtaining a second minus sign which cancels the first. And the Riemann integral $\int_{-b}^{-a} f(y) \, dy$ coincides with the Lebesgue integral $\int_{[-b,-a]} f(-y) \, d\lambda(y)$.

Another way to keep this straight in the context of Lebesgue integration is to remember that if f is a nonnegative function, then its integral over any set is nonnegative. Thus there can't possibly have been any missing minus sign, because $\int_{[-b,-a]} f(-y) \, d\lambda(y)$ is nonnegative whenever f is. □

Solution. This one is straightforward. □

Problem Set 11

For Friday May 7: Read §§5.0,2 (skip 5.1) of our text. Read §5.3 through the first sentence after the statement of Corollary 5.3.6. Solve the following problems: 5.0.4, 5.2.4,6 and 5.3.20,22,24.

Problems 5.2.6 and 5.3.24 have little to do with Lebesgue integration *per se*, but these are things which everyone ought to see. In problem 5.0.4, a function f is said to be *right-continuous* at a point $x \in \mathbb{R}$ if the one-sided limit $\lim_{y \to x^+} f(y)$ exists and equals $f(x)$. What's the point of this problem? If ν is a measure, the function $x \mapsto \nu([a, x])$ is automatically nondecreasing and right-continuous. This problem shows that this correspondence between measures and functions is bijective.

Now consider the Cantor ternary function F which we discussed in an earlier problem set. This function is continuous, is constant on each interval in the complement of the Cantor set $\mathcal{C}$ (the one whose measure is zero, defined by deleting middle thirds of intervals), and maps $\mathcal{C}$ onto $[0, 1]$. Via this problem it defines a certain measure ν on $\mathbb{R}$. By using the fact that F is constant on each interval in the complement of $\mathcal{C}$, one can deduce that $\nu(A) = 0$ whenever $A \cap \mathcal{C} = \emptyset$. Thus ν "lives" entirely on the Lebesgue null set $\mathcal{C}$. On the other hand, the continuity of F implies that $\nu(A) = 0$ for any finite set A.

(I hope that one or two lectures will remain after this material. If so, we'll discuss §6.1 on the inequalities of Jensen, Minkowski, Hölder, and Cauchy-Schwarz.)