## Mathematics 105 — Spring 2004 — M. Christ[1]

Solutions to selecta from problem set #1

**No number**. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that there exists some finite constant $M$ such that $|T(v)| \le M|v|$ for every $v \in \mathbb{R}^n$. (This was not assigned, but I used it in class and promised to provide a proof.)

**Solution.** Let $\{e_1, \cdots, e_n\}$ denote the standard basis for $\mathbb{R}^n$. Define the finite number $B = \max(|T(e_1)|, \cdots, |T(e_n)|)$. Consider any $v \in \mathbb{R}^n$ and write $v = (v^1, \cdots, v^n) = \sum_{j=1}^n v^j e_j$. Then

$$|T(v)| = |\sum_{j=1}^n v^j T(e_j)| \le \sum_{j=1}^n |v^j| \cdot |T(e_j)| \le B \sum_{j=1}^n |v^j|.$$

Now $|v^j| \le |v|$ for all $j$, so we find that $|T(v)| \le Bn|v|$. Thus it suffices to define $M = Bn$. (Actually $B\sqrt{n}$ would work, but there's no bonus for efficiency here.) $\square$

**I-4**. Prove that for any $x, y \in \mathbb{R}^n$, $\big||x| - |y|\big| \le |x - y|$.

**Solution.** First of all, if you tried to write this out using square roots and so on, you probably encountered a bit of a mess. A better way to do this is to remember that absolute values are often best reasoned about by cases, and to note that on the left-hand side of the inequality, the outermost symbols $\big|$ denote the absolute value of the real number $|x| - |y|$.

So split the proof into two cases. If $|x| \ge |y|$ then we're asked to show that $|x| - |y| \le |x - y|$. This is equivalent to $|x| \le |y| + |x - y|$. Since $|x| = |y + (x - y)|$, this is a direct consequence of the triangle inequality.

The problem is symmetric, in the sense that if the roles of $x, y$ are interchanged, then neither side of the inequality changes at all. Thus the case where $|y| \ge |x|$ is exactly the same as the case where $|x| \ge |y|$. $\square$

**I-14**. Prove that the union of any family of open sets is open. Prove that the intersection of any finite family of open sets is open, and give a counterexample showing that this need not hold for infinite families.

**Solution.** Let $S$ be some set and suppose that for each $s \in S$ there is given an open set $U_s \subset \mathbb{R}^n$. Let $A = \cup_s U_s$ be their union. Consider any point $a \in A$. Then there exists some $s$ for which $a \in U_s$. Since $U_s$ is open, there exists $r > 0$ such that the open ball $B(a, r)$ is contained in $U_s$. By definition of union, then, $B(a, r) \subset A$ as well. Thus $A$ is open. $\square$

Now suppose that $U_1, \cdots, U_N$ are open subsets of $\mathbb{R}^n$. Let $A = \cap_{j=1}^N U_j$, and consider any point $a \in A$. Then $a \in U_j$ for each $j$, so there exist radii $r_j > 0$ such that $B(a, r_j) \subset U_j$, for each $1 \le j \le N$. Define $r = \min_j r_j$; since this is a finite collection of positive numbers their minimum is well-defined and is strictly positive. Since $B(a, r) \subset B(a, r_j) \subset U_j$ for all $j$, we have $B(a, r) \subset A$. Thus $A$ is open. $\square$

For the counterexample, consider the open intervals $U_n = (-\frac{1}{n}, \frac{1}{n}) \subset \mathbb{R}^1$. These are open, but $\cap_{n=1}^\infty U_n = \{0\}$ is clearly not open. $\square$

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**I-16**. Find the interior, exterior, and boundary of each of the three sets $A_1 = \{x \in \mathbb{R}^n : |x| \leq 1\}$, $A_2 = \{x \in \mathbb{R}^n : |x| = 1\}$, $\mathbb{Q}^n = \{x = (x^1, \cdots, x^n) \in \mathbb{R}^n : x^i \in \mathbb{Q} \, \forall i\}$. ($\mathbb{Q}$ denotes the set of all rational numbers.)

**Solution.** I won't prove anything but will just give the answers. $A_1$: Its interior is the open ball $B(0,1)$. Its exterior is $\{x \in \mathbb{R}^n : |x| > 1\}$. Its boundary is $A_2$.

$A_2$: Its interior is the empty set. Its exterior is its complement, $\{x : |x| \neq 1\}$. Its boundary is itself.

$\mathbb{Q}^n$: Its interior is the empty set, its exterior is likewise the empty set, and its boundary is the entire space $\mathbb{R}^n$. $\square$

**I-21(b)**. Let $A, B$ be disjoint subsets of $\mathbb{R}^n$. Let $A$ be closed and let $B$ be compact. Prove that there exists $d > 0$ such that $|a - b| \geq d$ whenever $a \in A$ and $b \in B$.

**Solution.** By part (a) of this problem, we know that for each $b \in B$ there exists a positive number $r(b)$ such that $|a - b| \geq r(b)$ for all $a \in A$. We'd like to set $d = \min_{b \in B} r(b)$, but that doesn't make sense, since $B$ could be an infinite set and hence there might be no genuine minimum. We could try $\inf_{b \in B} r(b)$, but the infimum of a set of positive numbers might be zero.

Instead, consider the collection of open balls[2] $\{B(b, \frac{1}{2}r(b)) : b \in B\}$ (that is, form the collection of all open balls centered at arbitrary points $b \in B$, with radii $\frac{1}{2}r(b)$). This is an open cover of $B$, since $b \in B \Rightarrow b \in B(b, r(b))$.

Therefore there exists a finite subcover, that is, there exist finitely many points $b_1, \cdots b_N \in B$ such that $B \subset \cup_{j=1}^{N} B(b_j, \frac{1}{2}r(b_j))$. To simplify notation, define $r_j = r(b_j)$. Now define $d = \frac{1}{2}\min_{1 \leq j \leq N} r_j$. Being the minimum of finitely many positive numbers, $d$ is $> 0$.

Now let $b \in B$ be an arbitrary point, and let $a \in A$; we need to show that $|a - b| \geq d$. Since the collection of all balls $B(b_j, \frac{1}{2}r_j)$ is an open cover of $B$, there exists at least one index $i$ such that $b \in B(b_i, \frac{1}{2}r_i)$, that is, $|b - b_i| < \frac{1}{2}r_i$. By definition of $r(b_i)$, we also have $|a - b_i| \geq r_i$. Now invoke problem I-4:

$$|a - b| \geq |a - b_i| - |b - b_i| \geq r_i - \tfrac{1}{2}r_i = \tfrac{1}{2}r_i \geq d. \qquad \square$$

**I-21(c)**. Give an example of two closed sets in some $\mathbb{R}^n$ which do not satisfy the conclusion of part (b).

**Solution.** Let $A_1 = \{(x, y) \in \mathbb{R}^2 : y = 0\}$, and let $A_2 = \{(x, y) \in \mathbb{R}^2 : y = e^{-x}\}$. $A_2$ is the set of all points satisfying $f(x, y) = 0$, where $f : \mathbb{R}^2 \to \mathbb{R}^1$ is the continuous function $f(x, y) = y - e^{-x}$. Thus $A_2 = f^{-1}(\{0\})$, that is, $A_2$ is the inverse image of the closed set $\{0\}$ under the continuous function $f$. Therefore $A_2$ is closed. (It's also possible to prove this directly.) The same reasoning shows that $A_1$ is closed (consider $f(x, y) = y$).

Clearly $A_1, A_2$ are disjoint. Given any $d > 0$, choose $x \in \mathbb{R}^1$ sufficiently large that $e^{-x} < d$. Then the points $a = (x, 0) \in A_1$ and $a' = (x, e^{-x}) \in A_2$ satisfy $|a - a'| = e^{-x} < d$. Thus there exists no positive $d$ with the required property. $\square$

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[2] Standard notation: $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$.

**I-22**. Let $C \subset U \subset \mathbb{R}^n$ and suppose that $C$ is compact and $U$ is open. Show that there exists a compact set $D$ such that $C$ is contained in the interior of $D$, and $D \subset U$.
**Solution.** Let $A = \mathbb{R}^n \setminus U$ be the complement of $U$; $A$ is closed since $U$ is open. By problem 21(b), there exists $d > 0$ such that $|x - a| \geq d$ whenever $x \in C$ and $a \in A$. Consider the collection of all balls $B(x, \frac{1}{2}d)$ such that $x \in C$. This collection is an open cover of $C$, so there exists some finite subcover $\{B(x_j, \frac{1}{2}d) : 1 \leq j \leq N\}$.

Set $\tilde{D} = \cup_{j=1}^{N} B(x_j, \frac{1}{2}d)$. This is a finite union of open sets, so is open. It is contained in $U$, since each ball $B(x_j, d)$ is contained in $U$. Define $D$ to be the union of the *closures* of these same balls, that is, $D = \cup_j \{y : |y - x_j| \leq \frac{1}{2}d\}$. $D$ is a finite union of closed, bounded sets, so is closed and bounded, hence compact. $D \subset U$ for the same reason given above for $\tilde{D}$. $\tilde{D}$ is an open subset of $D$, so is contained in the interior of $D$. Thus the interior of $D$ contains $C$, and the proof is complete. □

**I-29**. Let $A \subset \mathbb{R}^m$ and let $f : A \to \mathbb{R}$ be continuous. Prove that $f$ is bounded, and takes on its maximum and minimum values.

*Note.* I will modify the problem statement by adding the hypothesis that $A$ is nonempty; a discussion about whether or not a continuous function defined on the empty set takes on its maximum value might be amusing but wouldn't be productive.
**Solution.** To show that $f$ is bounded above consider $V_n = \{a \in A : f(a) < n\}$, for each integer $n$. Since $f$ is continuous, there exists an open set $U_n \subset \mathbb{R}^m$ such that $V_n = U_n \cap A$.

If $a \in A$ then there exists $n$ such that $f(a) < n$, so $a \in V_n \subset U_n$. Thus the collection of sets $U_n$ forms an open cover of $A$; there exist finitely many integers $n_j$ such that $A \subset \cup_j U_{n_j}$. Since $U_n \cap A = V_n$, this means that $A = \cup_j V_{n_j}$. Thus any $a \in A$ satisfies $f(a) < \max_j n_j$, so $f$ is bounded above. The same reasoning shows that $f$ is bounded below (or apply the result just proved to $-f$).

Now we prove that $f$ actually takes on its maximum value. We know that $f$ is bounded. Consider the set $f(A) = \{f(a) : a \in A\}$. This set is nonempty since $A$ is nonempty, and is bounded, so it has a least upper bound in $\mathbb{R}^1$, which we call $y$. For each positive integer $n$ define $V_n = \{a \in A : f(a) < y - \frac{1}{n}\}$. Again $V_n = A \cap W_n$ for some open subset $W_n$ of $\mathbb{R}^m$.

Argue now by contradiction. If $a \in A$ and $f(a) < y$ then $f(a) < y - \frac{1}{n}$ for some $n$, so $a \in V_n$. Thus if $f$ never takes on the value $y$, then the family of open sets $W_n$ covers $A$. Therefore there is a finite subcover, and as above, this means that finitely many of the sets $V_n$ cover $A$. These sets $V_n$ are nested; if $n < k$ then $V_n \subset V_k$. Thus there exists a single set $V_N$ which covers $A$, and hence $f(a) < y - \frac{1}{N}$ for all $a \in A$. Thus means that $y - \frac{1}{N}$ is an upper bound for the range of $f$, contradicting the definition of $y$ as the *least* upper bound. Therefore $f$ must actually take on the value $y$.

Since the minimum of $f$ is $-1$ times the maximum of $-f$, by applying this result to $-f$ we conclude that $f$ also takes on its minimum value. □