

**Mathematics 105 — Spring 2004 — M. Christ**  
**Problem Set 9**

For Friday April 16: Continue to study §3.3 of our text. We will treat §3.4 in a somewhat superficial way by discussing the statement (3.4.7) *but not its proof*, and showing how it implies Theorem 3.4.8 and the related inequality (3.4.1). Theorem 3.4.8 expresses a nonobvious and fundamental property of integrable functions. Please read §3.4 in this spirit; you are very welcome to study the proof of (3.4.7) and the statement and proof of the rising sun (aka sunrise) lemma, but these are not officially part of this course and you will not be examined on them.

After this abbreviated discussion of §3.4 we will move on to §4.1 and then §5.1.

Solve the following problems from Stroock §3.3: 3.3.21 parts (i),(ii),(iii). 3.3.22 (By a finite measure space Professor Stroock means that  $\mu(E) < \infty$ , not that  $E$  is a finite set. The symbol “ $x \wedge y$ ” means “ $\min(x, y)$ ”, the minimum of  $x, y$ .) Solve the following problems.

**IX.A** Consider the measure space  $(\mathbb{R}^1, \overline{\mathcal{B}}_{\mathbb{R}^1}, \lambda)$  where  $\lambda$  denotes Lebesgue measure. Consider the measurable functions  $f_n(x) = \frac{1}{n}\chi_{[0,n]}$ . Show that  $f_n \rightarrow 0$  uniformly on  $\mathbb{R}$ . Show that  $\int f_n d\lambda \rightarrow 1$ . Explain why this does not contradict any of our three basic convergence theorems.

So far in the course we’ve learned some general theory about measure spaces. We’ve encountered two concrete kinds of measures, namely Lebesgue measure on  $\mathbb{R}^n$ , and counting measure ( $\mu(A) =$  the cardinality of  $A$  if  $A$  is a finite set, and  $\mu(A) = +\infty$  otherwise). We’ve learned that if  $(E, \mathcal{A}, \mu)$  is any measure space and  $f \in L^1$  is nonnegative, then a new measure  $\nu$  (for the same pair  $(E, \mathcal{A})$ ) can be constructed via the recipe  $\nu(A) = \int_A f d\mu$ . The next problem describes another way to construct new measures. (In Chapter 4 we’ll discuss yet another method, the formation of product measures, which is to measure spaces as the formation of Cartesian products is to sets.)

**IX.B** Let  $(E, \mathcal{A}, \mu)$  be any measure space. Suppose that  $f : E \rightarrow \mathbb{R}^1$  is a measurable function (which never takes on the values  $\pm\infty$ ). Recall that for any Borel measurable set  $S \subset \mathbb{R}^1$ , the set  $f^{-1}(S)$  belongs to  $\mathcal{A}$ . Define  $\nu(S) = \mu(f^{-1}(S))$  for all  $S \in \mathcal{B}_{\mathbb{R}^1}$  (You need not prove this.). Show that  $\nu$  is a measure on the Borel  $\sigma$ -algebra  $\mathcal{B}_{\mathbb{R}}$ .

More generally, suppose that  $f : E \rightarrow \mathbb{R}^n$  is a measurable mapping, in the sense that  $f^{-1}(S) \in \mathcal{A}$  for all  $S \in \mathcal{B}_{\mathbb{R}^n}$ . Show that the above recipe defines a measure on  $\mathcal{B}_{\mathbb{R}^n}$ .

**IX.C This problem as originally written was more challenging than I intended. It has been modified.** Recall the fat, or generalized, Cantor set  $\mathcal{C}_\alpha$  defined by starting with  $[0, 1]$ , deleting a subinterval centered at its midpoint of length  $\alpha/3$ , then deleting from each of the resulting two intervals centered subintervals of lengths  $\alpha/3^3$ , and so forth. Show directly (that is, using properties of  $\mathcal{C}_\alpha$  but not using the theorem) that the characteristic function of  $\mathcal{C}_\alpha$  satisfies the conclusion of Theorem 3.4.8 at every point of the **complement** of  $\mathcal{C}_\alpha$ . Show that for any point  $x \in \mathcal{C}_\alpha$ , there exists a sequence of intervals  $I_n$  containing  $x$  such that  $x \in I_n$ ,  $|I_n| \rightarrow 0$  as  $n \rightarrow \infty$ , and  $|I_n \cap \mathcal{C}_\alpha|/|I_n| \rightarrow 1$  as  $n \rightarrow \infty$ . (The purpose of this exercise is to help you to understand what Theorem 3.4.8 says. To appreciate how nonobvious it is, try proving directly that its conclusion holds at almost every  $x \in \mathcal{C}_\alpha$ .)

**IX.D** In any measure space  $(E, \mathcal{A}, \mu)$  satisfying  $\mu(E) < \infty$  let  $\{f_n\}$  be a sequence of measurable functions, which converge almost everywhere to a limit  $f$ . Egoroff’s theorem says that for any  $\varepsilon > 0$ , there exists a measurable set  $A$  satisfying  $\mu(E \setminus A) < \varepsilon$  such that  $f_n \rightarrow f$  *uniformly* on  $A$ . Prove this. (Outline: Define bad sets  $B_k = \{x : \exists n \geq N_k \text{ such that } |f_n(x) - f(x)| > 2^{-k}\}$ . Choose  $N_k$  sufficiently large that  $\mu(B_k) < 2^{-k}\varepsilon$ ; note that the hypothesis  $\mu(E) < \infty$  implies that  $f_n \rightarrow f$  *in measure*, whence  $N_k$  exists. Define  $A$  to be the complement of  $\cup_{k=1}^\infty B_k$ .)

Thus in any universe of finite measure, almost everywhere convergence implies uniform convergence, except for an arbitrarily small exceptional set of points.

**IX.E** Use Egoroff’s theorem to give an alternative proof of the bounded convergence theorem: If  $\mu(E) < \infty$ , if  $f_n \rightarrow f$   $\mu$ -a.e. on  $E$ , if  $f_n$  are all measurable, and if there exists  $M < \infty$  such that  $|f_n(x)| \leq M$  for all  $n$  and all  $x \in E$ , then  $\int_E f_n d\mu \rightarrow \int_E f d\mu$ .