Mathematics 105 — Spring 2004 — M. Christ
Problem Set 8 (corrected[1])

For Friday April 9: Study §3.3 of our text. Solve the following problems from Stroock §3.3: 3.3.16,17,19,20,23.

In problem 23, simplify the statement by assuming that $\mu(E)$ and $\nu(E)$ are both finite. Thus it is given that $\int \varphi\, d\mu = \int \varphi\, d\nu$ for *all* bounded, ρ-uniformly continuous functions φ, and you are asked to prove that $\mu = \nu$ on $\mathcal{B}_E$. (Recall that the Borel sigma-algebra $\mathcal{B}_E$ is the smallest sigma-algebra of subsets of E which includes all open subsets of E. Recall also that we have discussed the universal method of proving statements about Borel sets; that discussion applies to the metric space (E, ρ) just as well as to $\mathbb{R}^n$.)

In problem 3.3.22, in the phrase "whenever φ is a bounded ρ-uniformly continuous φ", the second "φ" should be replaced by the word "function".

There's a typo in the second-to-last line of exercise 3.3.16; it should read "any sequence of $\mathcal{B}$-measurable sets". Our author just wants to say that $\Gamma_n \in \mathcal{B}$.

Hint for 3.3.16: Prove it first for bounded f, then approximate a general unbounded f by bounded functions.

VIII.A Let $f : [a, b] \times [c, d] \to \mathbb{R}$ and suppose that for each $y \in [c, d]$, $[a, b] \ni x \mapsto f(x, y)$ is a measurable function. Suppose also that the partial derivative $\partial f/\partial y$ exists for all $(x, y) \in [a, b] \times (c, d)$, and that its absolute value is bounded by a certain finite constant, which is independent of x, y. Prove that for each $y \in (c, d)$, $x \mapsto \partial f(x, y)/\partial y$ is a measurable function of $x \in [a, b]$. Prove that for all $y \in (c, d)$,

$$\frac{d}{dy} \int_{[a,b]} f(x, y)\, d\lambda(x) = \int_{[a,b]} \frac{\partial f(x, y)}{\partial y}\, d\lambda(x).$$

Comment. Even if f and $\partial f/\partial y$ are continuous functions, so that both integrals can be regarded as Riemann integrals, this result is superior to the standard one which arises naturally in the theory of Riemann integration, namely, that the conclusion holds provided that the difference quotients defining $\partial f/\partial y$ converge uniformly (as functions of (x, y)) to that partial derivative.

This type of result has plenty of applications.

VIII.B Let $(E, \mathcal{A}, \mu)$ be any measure space and let $f \in L^1(E, \mu)$. Consider the mapping $L_f(A) = \int_A f\, d\mu$, where the domain is the collection of all sets $A \in \mathcal{A}$. Give an example to show that, for general measure spaces, even if $\|f\|_{L^1} \le 1$, the parameter δ in problem 3.3.16 cannot be chosen to depend on ε and $\|f\|_{L^1}$ alone.

[1] Oops — I was looking at a copy of the second edition of the text when I originally chose the problems.