Mathematics 105, Spring 2004 — Problem Set VII

Remark on problem 3.1.11: If (x_n) is a sequence of real numbers, $\overline{\lim}_{n\to\infty} x_n$ is a common alternative notation for $\limsup_{n\to\infty} x_n$; likewise $\underline{\lim}$ is denotes the lim inf. In this problem analogous notions of lim sup and lim inf are defined for sequences of sets. Inequality (i) deals with a relation between the lim inf of the real numbers $\mu(\Gamma_n)$ and the μ measure of the lim inf of the sets Γ_n; thus the first symbol $\underline{\lim}$ refers to the notion for sets, while the second instance of the same symbol refers instead to the notion for real numbers.

A couple of other miscellaneous remarks on the text:

(i) The discussion in §§3.1 and 3.2 is a bit more abstract and general than is really necessary for us. In class I'll discuss the main elements, in a streamlined way. I'll be perfectly satisfied if you master the topics discussed in lecture. One example is the concept of a measurable mapping. We'll simply discuss measurable $\overline{\mathbb{R}}$-valued functions, that is, measurable mappings from some measure space to the extended real numbers $\overline{\mathbb{R}} = [-\infty, +\infty]$, and I'll give a more direct definition. For present purposes we won't need to know about tensor products of measure spaces or of measurable mappings (this will come up later, when we discuss the connection between Lebesgue measure/integration on $\mathbb{R}^n, \mathbb{R}^m$, and $\mathbb{R}^{m+n}$.

(ii) Wherever Stroock mentions topological spaces, you should assume that he is talking about *metric* spaces (which are a special class of topological spaces); for our purposes metric spaces are sufficiently general.

(iii) For Stroock, $\restriction$ is a general notation for restriction. Thus $f \restriction E$ denotes the restriction of a function f to a subset E of its domain, and there are various other related usages.

For Friday March 19: Solve problems Stroock #3.1.14, #3.2.17, and the five problems formulated below. (I quite like #3.1.14, which establishes a surprising concrete fact about probability and permutations.) (Further hint for #3.2.17: Show that if $c > 0$ and if $\varphi(x) \ge c$ for every $x \in E$, and if $\mu(E) > 0$, then $\int_E \varphi \, d\mu > 0$. Then use this to show that if merely $\varphi(x) > 0$ for every $x \in E$, then the same conclusion holds.)

VII.A In this problem we construct a continuous function $f : [0,1] \to \mathbb{R}$ which has some surprising properties. The construction is based on the Cantor set $\mathcal{C}$ constructed in Stroock's problem 2.1.20. Recall that $\mathcal{C} = \cap_{k=0}^\infty \mathcal{C}_k$ where $\mathcal{C}_k$ is the union of 2^k closed intervals $\{I_j^k : 1 \le j \le 2^k\}$, each of which has length 3^{-k}, and $\mathcal{C}_{k+1}$ is the subset of $\mathcal{C}_k$ obtained by deleting the (open) middle third of each interval I_j^k. Thus $\mathcal{C}$ has Lebesgue measure equal to zero. Recall also that the open set $[0,1] \setminus \mathcal{C}$ is a disjoint union of open intervals, the sum of whose lengths equals one.

Let's write $I_j^k = [a_j^k, b_j^k]$, and order them from left to right, so that $a_1^k = 0$, $b_j^k < a_{j+1}^k$ for all j, and $b_{2^k}^k = 1$. Note that then $a_{j+1}^k - b_j^k = 3^{-k}$. The open intervals mentioned above are all the intervals (b_j^k, a_{j+1}^k), where k, j range over all allowed values. Define a sequence of functions $f_k : [0,1] \to [0,1]$ as follows: f_k is the unique continuous piecewise-linear function which maps 0 to 0, has constant slope $(3/2)^k$ on each interval I_j^k, and is constant on the interval $[b_j^k, a_{j+1}^k]$ for all $1 \le j < 2^k$.

(i) Prove that the sequence $\{f_k\}$ converges uniformly to some continuous function $f : [0,1] \to [0,1]$. (Hint: To get convergence, show that the sequence $\{f_k(x)\}$ is Cauchy for all x by finding an upper bound for $|f_k(x) - f_{k+1}(x)|$. Then prove that your bound also implies uniform convergence.) (ii) Show that f is constant on each interval $[b_j^k, a_{j+1}^k]$. Conclude that $f([0,1] \backslash \mathcal{C})$ is countable, hence has measure zero. (iii) Observe that $f(0) = 0$ and $f(1) = 1$. Show that f maps the interval $[0,1]$ onto itself. Conclude that $|f(\mathcal{C})| = 1$, even though $|\mathcal{C}| = 0$. (iv) Conclude that $\mathcal{C}$ is uncountable. (In problem 2.1.20 this was proved in a different way.) (v) Define $g(x) = f(x) + x$, and note that $g(0) = 0$ and $g(1) = 2$. Show that g is *strictly* increasing, that is, if $0 \le x_1 < x_2 \le 1$ then $g(x_1) < g(x_2)$. Thus g is a homeomorphism from $[0,1]$ to $[0,2]$; it is a bijective continuous function with a continuous inverse. (vi) Show that g maps $\mathcal{C}$ onto a compact set whose Lebesgue measure equals 1. Thus even a homeomorphism can map a set of measure zero onto a set of positive measure.

VII.B Let E be a set, and let $\mathcal{A}$ be a σ-algebra of subsets of E. A function $f : E \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$, is defined to be *measurable* if for any $a \in \mathbb{R}$, $\{x \in E : f(x) > a\}$ belongs to $\mathcal{A}$. (This is the definition I gave in class; Stroock gives an equivalent definition which looks more complicated.)

(a) Prove that all of the following are mutually equivalent. (i) f is measurable. (ii) For any $a \in \mathbb{R}$, $\{x \in E : f(x) \ge a\}$ belongs to $\mathcal{A}$. (iii) For any $a \in \mathbb{R}$, $\{x \in E : f(x) < a\}$ belongs to $\mathcal{A}$. (iv) $\{x \in E : f(x) = +\infty\}$ and $\{x \in E : f(x) = -\infty\}$ both belong to $\mathcal{A}$, and moreover for any open set $G \subset \mathbb{R}$, $f^{-1}(G) \in \mathcal{A}$.

(b) Prove that if f is measurable, then for any $a \in \mathbb{R}$, $f^{-1}(\{a\}) = \{x : f(x) = a\}$ is measurable. Give an example to show that there also exist nonmeasurable functions that satisfy this conclusion.

VII.C Let $\{f_n\}$ be a sequence of measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$. Let $L = \{x : \lim_{n\to\infty} f(x)$ exists and is finite$\}$. Show that L is measurable. (Outline: (i) Show that $\limsup_{n\to\infty} f_n(x)$ is a measurable function. Note that since $\liminf f_n = -\limsup(-f_n)$, it follows that the lim inf is likewise measurable; you may use this in part (ii). (ii) Use the fact that the limit exists and is finite if and only if the lim sup and lim inf are both finite and $\limsup f_n(x) - \liminf f_n(x) = 0$.)

VII.D In class I showed that the sum of two measurable functions is measurable (provided they satisfy a small extra hypothesis). Use the same reasoining to prove that for any two measurable functions $f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$, the product $fg = f \cdot g$ is measurable. (Here "measurable" means with respect to the sigma-algebra $\overline{\mathcal{B}} = \overline{\mathcal{B}}_{\mathbb{R}^n}$.) (Unlike the case of sums, no extra hypothesis is needed here, because the product of any two elements of $\overline{\mathbb{R}}$ is defined.)

VII.E Give an example of a measurable function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every $x \in \mathbb{R}$. (You need not give a detailed proof.)