

## Mathematics 105, Spring 2004 — Problem Set IV

For the remainder of the semester we will follow the text *A Concise Introduction to the Theory of Integration* (second edition) by Stroock. Problem, chapter, section, and page numbers refer to that text unless otherwise indicated.

For Friday February 27: You should have already read §2.0 and begun reading §2.1. Finish studying §2.1. Also read Lemma 1.1.1 and its proof; the whole theory of Lebesgue integration relies on this (semitrivial) lemma.

Solve problems 2.1.18,19 of Stroock, and the following problems:

**IV.A** Let  $\{I_n\}$  be any *finite* set of open intervals that covers  $[0, 1] \cap \mathbb{Q}$ . Show that  $\sum_n |I_n| \geq 1$ . Explain why this does *not* prove that  $|[0, 1] \cap \mathbb{Q}|_e \geq 1$ .

**IV.B** Show that for any Lebesgue measurable sets  $A, B \subset \mathbb{R}^n$ ,  $|A \cup B| + |A \cap B| = |A| + |B|$ .

**IV.C** Let  $A \subset \mathbb{R}^n$  and suppose that  $|A|_e = 0$ . Prove that for any set  $B \subset \mathbb{R}^n$ ,  $|A \cup B|_e = |B|_e$ .

**IV.D** Let  $A \subset \mathbb{R}^1$  be the set constructed in Lemma 2.1.16. Prove that for any measurable set  $B \subset A$ ,  $|B| = 0$ .

**IV.E** Let the sequence  $(x_n)_{n=1}^\infty$  be an enumeration of  $(0, 1) \cap \mathbb{Q}$ . (That is, for each  $q \in (0, 1) \cap \mathbb{Q}$  there exists a unique  $n \in \mathbb{N}$  such that  $q = x_n$ .) Define the open intervals  $I_n = (x_n - \delta_n, x_n + \delta_n)$  where  $\delta_n = \frac{1}{10}2^{-n}$ .

Let  $G = \bigcup_{n=1}^\infty I_n$ , and let  $E = G \cap (0, 1)$ . (a) Show that  $E$  is a dense open subset of  $(0, 1)$ . ( $E$  is said to be dense in  $(0, 1)$  if for each  $y \in (0, 1)$  and each  $\varepsilon > 0$  there exists  $x \in E$  such that  $|x - y| < \varepsilon$ .)

(b) Show that  $E \neq (0, 1)$ . (Hint: Use Lebesgue measure; find an upper bound for  $|E|$ .)

*Comment.* IV.E shows that open sets, even in dimension one, can be quite a bit more complicated than finite unions of intervals.

*Review.* It is possible to enumerate the elements of  $\mathbb{Q} \cap (0, 1)$ . (i) Consider the sequence  $\frac{0}{1}, \frac{1}{1}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{0}{4}, \dots$ . Every element of  $\mathbb{Q} \cap [0, 1]$  appears in this list, and appears infinitely many times. (ii) Go through the list (from left to right), and cross out any number which does not belong to the open interval  $(0, 1)$ , and cross out any number which is not expressed in lowest terms (that is, with the smallest possible denominator). Thus  $\frac{0}{1}, \frac{1}{1}, \frac{0}{2}, \frac{2}{2}, \dots$  get crossed out, and  $\frac{2}{4}$  gets crossed out because it equals  $\frac{1}{2}$ ,  $\frac{2}{6}, \frac{3}{6}, \frac{4}{6}$  get crossed out because they equal  $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ , and so on. Because each positive rational has a unique expression  $\frac{p}{q}$  with minimal denominator  $q$ , each element of  $\mathbb{Q} \cap (0, 1)$  then appears exactly once on this list.