Mathematics 105 — Spring 2004 — M. Christ
Problem Set 10

For Friday April 30: Continue to study §4.1 of our text.

Solve the following problems from Stroock §4.1: 4.1.8, 10, 11, 12. (For 4.1.10, note that the σ-algebras in question are the *Borel* algebras $\mathcal{B}_{\mathbb{R}^k}$ rather than the associated *Lebesgue* algebras $\overline{\mathcal{B}}_{\mathbb{R}^k}$.) For 4.1.11, the things you are asked to show in the sentence "Solve also the following problems ..." were (or will be) shown in class; you need not reprove them but should just prove what is asked in the sentence beginning "Finally".

X.A Here are two associative laws: Let $(E_j, \mathcal{A}_j, \mu_j)$, for $j = 1, 2, 3$, be three σ-finite measure spaces. Prove that the two σ-algebras $(\mathcal{A}_1 \times \mathcal{A}_2) \times \mathcal{A}_3$ and $\mathcal{A}_1 \times (\mathcal{A}_2 \times \mathcal{A}_3)$ (both of which are σ-algebras of subsets of $E_1 \times E_2 \times E_3$) are equal. Prove that the two measures $(\mu_1 \times \mu_2) \times \mu_3$ and $\mu_1 \times (\mu_2 \times \mu_3)$ on $E_1 \times E_2 \times E_3$ are equal.

X.B Another example: Let $(E_j, \mathcal{A}_j, \mu_j)$ equal $\mathbb{N}$ equipped with the σ-algebra of all its subsets, and with counting measure. (Thus $\mathcal{A}_1 \times \mathcal{A}_2$ consists of all subsets of $\mathbb{N} \times \mathbb{N}$.) Define $f(x_1, x_2)$ to be 1 if $x_1 = x_2 \in \mathbb{N}$, to be -1 if $x_2 = x_1 + 1$, and to be 0 otherwise. Show that $\iint f \, d\mu_1 \, d\mu_2 \neq \iint f \, d\mu_2 \, d\mu_1$. Which hypothesis of Fubini's theorem is not satisfied?

X.C As a complement to problem 4.1.10, show that if $(E_j, \mathcal{A}_j, \mu_j) = (\mathbb{R}^1, \overline{\mathcal{B}}_{\mathbb{R}^1}, \lambda)$ where λ denotes Lebesgue measure on $\mathbb{R}^1$, then the product measure $\mu_1 \times \mu_2$ on $\mathcal{A}_1 \times \mathcal{A}_2$ is *not* complete. (Hint: Consider $\{0\} \times A$ where $A \subset \mathbb{R}^1$ is not measurable.)

X.D Earlier in the course we sweated quite a bit to prove that $|T(A)| = |\det(T)| \cdot |A|$ for any measurable set $A \subset \mathbb{R}^n$ and any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. One part of this argument can now be reworked as a consequence of Fubini's Theorem (and problem 4.1.10), as follows. (For simplicity I'll restrict the problem to Borel sets; the extension to Lebesgue measurable sets follows fairly easily from the fact that any such set is the union of a Borel set with a set of measure zero. Likewise I'll simplify the discussion by assuming T to be invertible.)

We take for granted the fact that any linear transformation of $\mathbb{R}^n$ can be expressed as a finite composition of linear transformations of three basic types[1]: (i) Dilations $x = (x_1, \cdots, x_n) \mapsto (cx_1, x_2, \cdots, x_n)$ for some $c \neq 0$; (ii) Permutations $x \mapsto (x_k, \cdots, x_{k-1}, x_1, x_{k+1}, \cdots, x_n)$; and (iii) Shear transformations $x \mapsto (x_1, \cdots, x_{k-1}, x_k + cx_1, x_{k+1}, \cdots, x_n)$. (In (ii) and (iii), $k \in \{1, 2, \cdots, n\}$ is arbitrary.) Let's take for granted the result for linear transformations of the first two types, and let's simplify the notation by discussing only dimension $n = 2$.

Show that for any shear transformation T of $\mathbb{R}^2$, $|T(A)| = |A|$ for all Borel subsets A of $\mathbb{R}^2$. (Use Fubini's theorem, not Theorem 2.2.2!) Explain briefly why this leads to the conclusion $|T(A)| = |\det(T)| \cdot |A|$ (for all Borel subsets A of $\mathbb{R}^2$), for all invertible linear transformations T of $\mathbb{R}^2$.

X.E We have studied the set $L^1 = L^1(\mathbb{R}^n, \overline{\mathcal{B}}_{\mathbb{R}^n}, \lambda)$ of all Lebesgue measurable, integrable functions from $\mathbb{R}^n$ to the extended real numbers, and have learned that this set is a complete normed linear space when equipped with the L^1 norm. In this problem we will learn that this set of functions actually has the structure of an *algebra*.[2] You might guess that the algebra structure would be defined by pointwise multiplication, the product of f, g being

[1] This just amounts to the fact that any invertible matrix can be reduced to the identity matrix via the usual row and column operations.

[2] This algebra structure arises naturally in many applications, for instance in the analysis of the basic partial differential equations of mathematical physics.

$h(x) = f(x)g(x)$, but it turns out (see part (a) below) that this pointwise product of two L^1 functions need not belong to L^1. Instead, the product is defined formally by

$$f * g(x) = \int f(x-y)g(y)\, d\lambda(y) \tag{1}$$

where λ denotes Lebesgue measure on $\mathbb{R}^n$.

X.E(a) Let f be defined on $\mathbb{R}^1$ by $f(x) = x^{-1/2}$ for all $x \in (0,1])$ and $= 0$ for all other x. Show that $f \in L^1$, but the function $x \mapsto f(x)^2$ does not belong to L^1.

X.E(b) In the rest of the problem we fix a dimension n and use both x and y for coordinates on $\mathbb{R}^n$. Thus (x, y) denotes a point in $\mathbb{R}^n \times \mathbb{R}^n$, which we identify with $\mathbb{R}^{2n}$ in the natural way. Show that if $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is *Borel* measurable, then the function $F(x, y) = f(x-y)$ is a Borel measurable function on $\mathbb{R}^{2n}$. (Hint: The map $(x, y) \mapsto x - y$ is continuous; the inverse image of any Borel set under any *continuous* mapping is a Borel set.)

X.E(c) Show that if $A \subset \mathbb{R}^n$ is Lebesgue measurable with measure 0, then $\tilde{A} = \{(x, y) \in \mathbb{R}^{2n} : x - y \in A\}$ has exterior measure zero, and therefore is Lebesgue measurable. (Hint: It suffices to show that for any $R < \infty$, $\tilde{A} \cap S_R$ has exterior measure zero, where $S_R = \{(x, y) : |y| < R\}$. Take any open set G containing A of measure $< \varepsilon$, consider $\tilde{G} \supset \tilde{A}$, note that $\tilde{G}$ is (Borel) measurable by part (a), use Tonelli's theorem to get an upper bound for $|\tilde{G} \cap S_R|$ in terms of R, ε, and let $\varepsilon \to 0$ with R held fixed.)

X.E(d) Deduce from the preceding that if $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is Lebesgue measurable, then the function defined by $F(x, y) = f(x - y)$ is Lebesgue measurable.

X.E(e) For the remainder of the problem assume that $f, g \in L^1(\mathbb{R}^n)$. Show[3] that the function $h(x, y) = f(x - y)g(y)$ is Lebesgue measurable (on $\mathbb{R}^{2n}$, of course).

X.E(f) Show further that $h \in L^1(\mathbb{R}^{2n})$. (Hint: Use Tonelli's theorem, noting that $\int\int |f(x-y)g(y)|\, d\lambda(x)\, d\lambda(y)$ is very easy to evaluate. You may use without proof the simple change of variables formula $\int_{\mathbb{R}^n} f(x-z)\, d\lambda(x) = \int_{\mathbb{R}^n} f(t)\, d\lambda(t)$.) Show as a corollary that the product $f(x-y)g(y)$ is well-defined for almost every $(x, y) \in \mathbb{R}^{2n}$, and that this defines a measurable function of (x, y).

X.E(g) Show that for almost every $x \in \mathbb{R}^n$, the function $x \mapsto f(x - y)g(y)$ belongs to $L^1(\mathbb{R}^n)$, and moreover that $y \mapsto \int f(x-y)g(y)\, d\lambda(y)$ defines a measurable function in $L^1(\mathbb{R}^n)$. Thus we've shown that (1) does define an element of $L^1(\mathbb{R}^n)$, for arbitrary $f, g \in L^1(\mathbb{R}^n)$. (You are of course permitted to invoke Fubini's theorem!)

X.E(h) Show that for any $f, g \in L^1(\mathbb{R}^n)$, $\int f * g\, d\lambda = \int f\, d\lambda \cdot \int g\, d\lambda$, and that $\|f * g\|_{L^1} \leq \|f\|_{L^1} \cdot \|g\|_{L^1}$.

X.E(i) Show that $f * g = g * f$ (in the almost-everywhere sense). (You are permitted to use without proof any reasonable change-of-variables formula that you need to derive this.) Similar reasoning shows that the product $*$ satisfies the associative law, but you're not asked to prove this.

X.E(j) Show that for any three functions in $L^1(\mathbb{R}^n)$ and any $a, b \in \mathbb{R}$, $f * (ag + bh) = af * g + bf * h$. All together, this proves that the convolution product (1) does define the structure of an *algebra* on $L^1(\mathbb{R}^n)$.

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[3] In an earlier draft I wrote $|f(x-y)g(y)|$ only because I stupidly forgot that in this course, we've agreed that the product of any two extended real numbers is defined; I was being ultra-cautious by excluding from consideration products of plus infinity with minus infinity.