Mathematics 105, Spring 2004 — M. Christ
Midterm Exam #2 Comments

Distribution of scores: There were 50 points possible. The highest score was 50 and the third highest was 39. The median was $25\frac{1}{2}$, the 75th percentile was 35, and the 25th percentile score was 21. Final course grades will be based[1] 15% on MT1, 20% on this exam, 20% on problem sets, and 45% on the final exam.

(1a) In defining a Lebesgue measurable set, a few students wrote $|G \setminus A| < \varepsilon$ instead of $|G \setminus A|_e < \varepsilon$. The distinction is that $|G \setminus A|$ is not defined unless one already knows that $G \setminus A$ is measurable. In testing measurability, one uses exterior measure, which is defined for all sets.

(1b) If f is measurable then $\int_E f\,d\mu$ is defined if and only if the two numbers $\int_E f^+\,d\mu$ and $\int_E f^-\,d\mu$ are not *both* equal to $+\infty$.

A few students also mentioned the case where both integrals equal $-\infty$. But this can never arise since f^+, f^- are by definition *nonnegative* functions.

The question did not call for a *definition* of $\int_E f\,d\mu$, so there was no need for you to reproduce the chain of definitions.

(1c) A few people forgot to put absolute value signs around $\det(T)$. Clearly these are needed; the Lebesgue measure of $T(A)$ can't be negative!

(2a) The relevant example is the "fat" or generalized Cantor set, which we discussed in a homework problem. Given $\alpha \in (0,1)$, this set is constructed by deleting from $[0,1]$ a subinterval of length $\alpha/3$, leaving two subintervals of equal lengths. From the middle of each of those we delete a subinterval of length $\alpha/3^2$, leaving four subintervals of equal lengths. From the middle of each of those we delete a subinterval of length $\alpha/3^3$, and so forth; the fat Cantor set is the set of all points which are never deleted if this process is continued through infinitely many steps.

(2b) $\mathbb{Q} \cap [0,1]$ is an example, as was shown in a homework problem. It is an essential part of the definition of outer (exterior) measure, and hence of Lebesgue measure, that (countably) infinite covers are allowed; using finite covers by rectangles to define outer measure would yield a different notion under which, for instance, $\mathbb{Q} \cap [0,1]$ would have outer measure equal to 1 rather than equal to 0.

(3) Please see Stroock or lecture notes for solution. The most common problem here (and it turned out to be quite a bit more common than I had expected) was a misunderstanding of the relationship between $S = \{x : f(x) > a\}$ and the sets $S_n = \{x : f_n(x) > a\}$, given that $f_n(x) \to f(x)$ for all $x \in E$. A typical error was to assert that $S = \cup_n S_n$ or $= \cap_n S_n$. To refute the first assertion consider the example $f_1(x) = 100$ and $f_n(x) = -1$ for all $n \geq 2$, with $a = 0$. To refute the second set $a = 0$ and $f_n(x) = 1/n$ for all n. Then $x \in S_n$ for all n but $f(x) = 0$ so $x \notin S$.

A few people complicated these assertions by introducing sets like $\cup_n \cup_{q \in \mathbb{Q}^+} \{x : f_n(x) > a + q\}$, without affecting the essential difficulty described above.

A good method of attack was to note that $\lim_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)$ for all x (there's no need to bring the lim inf into the discussion in this problem, since existence of the limit for all x is given). Setting $g_n(x) = \sup_{k\geq n} f_k(x)$, it is true that $\{x : g_n(x) > a\} = \cup_{k\geq n}\{x : f_k(x) > a\}$, so the set on the left is a countable intersection of measurable sets and hence is measurable. Now a subtler mistake was sometimes made, namely the assertion that $\{x : f(x) > a\} = \cap_n\{x : g_n(x) > a\}$. Here $g_n(x)$ decreases to $\limsup f_n(x) = f(x)$ for all x, by its definition. However it could happen that $g_n(x) > a$ for all n, but the limit $f(x)$ equals a.

A true statement is that $\{x : f(x) \geq a\} = \cap_n\{x : g_n(x) \geq a\}$, and this leads to a solution.

This problem illustrates nicely how lim sup and lim inf are often useful in discussing genuine limits; they're not merely substitutes to be used when the limit fails to exist.

[1] If you ever plan to be teaching and grading in your future career, you might find it instructive to reflect on what on earth this sentence could actually mean, particularly in light of concepts like standard deviation.

(4) This problem caused quite a bit more difficulty than I'd anticipated; I expected problem (5) to be the tricky one but on the average people had more trouble with number (4). Let's fix firmly in our minds the fact that this result is *false* if it is only given that $0 \le f_j(x) \to \chi_A(x)$ for all x. Indeed consider $f_j = \chi_{[j,j+1]}$ and $A = \emptyset$ (with $(E, \mathcal{A}, \mu) = (\mathbb{R}, \overline{\mathcal{B}}, \lambda)$). Then $0 \le f_j(x) \to \chi_A(x)$ for all x, yet $\int f_j \, d\lambda = 1$ while $|A| = 0$.

Another example is $f_j(x) = -1/j$ for all $x \in \mathbb{R}$. Now all hypotheses are satisfied with $A = \emptyset$, except that f_j are not nonnegative. Again the conclusion is false.

Many solutions introduced a bit of notation and then simply claimed the conclusion without offering any analysis at all. In particular, if you didn't use the monotonicity hypothesis $f_j \le f_{j+1}$ and the hypothesis that $f_j \ge 0$, then you can't possibly have writtenwritten a correct proof!

A more sophisticated fallacy, to which several victims fell prey, is to write f_j out in the form $f_j = \sum_{k=1}^{N_j} c_{j,k} \chi_{B_{j,k}}$ where $c_{j,k}$ are nonnegative scalars and the sets $B_{j,k}$ are pairwise disjoint for each fixed j as k varies and are measurable. (Often this was done with deficient notation which didn't involve a double subscript, and didn't acknowledge that the number N_j of terms in the j-th sum could depend on j and could even tend to infinity with j — a potential source of trouble in any proof.) It then follows directly that either $c_{j,k} = 0$, in which case this term may be dropped, or $B_{j,k} \subset A$.

The more sophisticatd error was then to assert that as $j \to \infty$, $\min_k c_{j,k}$ tends to 1; in other words, $f_j \to \chi_A$ <u>uniformly</u> as a function of $x \in E$ as $j \to \infty$. This does not follow from the hypotheses. For example, we could have (with $E = \mathbb{R}^1$) $f_j(x) = 0$ for all $x \le 1/j$ and $= 1$ for $x > 1/j$ so that $A = (0, \infty)$. The hypotheses are satisfied, yet the conclusion is not uniform. Worse, the number of terms N_j could tend to infinity, allowing worsifications of this basic example.

(5) Please see text for a solution. Some very good solutions of this longish problem were submitted. Other students proved beyond doubt that they had studied the proof, but produced only disconnected fragments rather than a coherent argument.

It's not possible to find an *exact* cover of any set by rectangles. For instance, it's impossible for a Cantor set of measure zero; in any exact cover by intervals, each interval would be forced to have length zero, hence would contain a single point, so such a "cover" would cover only countably many points, hence would miss most of the Cantor set.

What we proved is that this can be done for any *open* set; it can be covered by a nonoverlapping family of cubes lying inside it, the sum of whose measures equals the total measure. Therefore a correct proof begins by approximating A from the outside by an open set whose measure is nearly zero.

It's true, and immediate, that for any set S of diameter δ, $f(S)$ is contained in a set of diameter $\le 3\delta$. However some asserted implicitly or explicitly that for any rectangle R of diameter δ, $f(R)$ is contained in a *rectangle* whose outer measure is $\le C|R|$, where C is a certain finite constant whose exact value is not important here. This is false. If $f(x, y) = (x, y - x^2)$ for all (x, y) in some neighborhood of the origin, then f transforms a rectangle $R = [0, c] \times [0, \delta]$ to a set which is shaped, for small δ, like a very skinny banana (draw it to see). Let c remain fixed and imagine what happens as $\delta \to 0$. If you draw any rectangle containing $f(R)$, you find that its volume is bounded below by a positive constant which does not tend to zero with δ.