Mathematics 105, Spring 2004 — Midterm Exam #1 Solutions

(1b) Give an example of a function $f : \mathbb{R}^3 \to \mathbb{R}^1$ such that f is not differentiable at $a = (0, 0, 0)$, but all partial derivatives $D_i f(a)$ do exist.

Solution: Define $f(0, 0, 0) = 0$, and for all $x = (x^1, x^2, x^3) \neq (0, 0, 0)$ define $f(x) = x^1 x^2 x^3/|x|^2$. □

(Justification was not required, but here it is: All three partial derivatives $D_i f(0, 0, 0)$ vanish, yet other directional derivatives at $0 = (0, 0, 0)$ are not zero. On the other hand, according to one of our theorems, if f were differentiable at 0 then the Jacobian matrix $f'(0)$ would have to equal zero since its entries are the partial derivatives. Now according to a homework problem, all directional derivatives can be calculated from the derivative by $D_v f(0) = Df(0)(v)$. Thus all directional derivatives must also equal 0. But it's easy to see from the formula that for $v = (1, 1, 1)$, say, $D_v f(0)$ is $\neq 0$.)

(2b) Let V, W be two open subsets of $\mathbb{R}^n$, and suppose that $f : V \to W$ is an invertible function with inverse g. Suppose that f is differentiable at $a \in V$, and g is differentiable at $b = f(a)$. State and prove a formula relating $f'(a)$ to $g'(b)$.

Solution: Let $h(x) = x$ for all $x \in \mathbb{R}^n$. Then $g \circ f(x) = h(x)$ for all $x \in V$. Since f, g are differentiable at a and at $b = f(a)$, respectively, the chain rule says that $h'(a) = g'(b) \cdot f'(a)$. Now since h is a linear transformation, $Dh(a)(v) = h(v) = v$ for all $v \in \mathbb{R}^n$, by a theorem in our text. Thus $h'(a)$ is the identity matrix I, so $g'(b) \cdot f'(a) = I$. Thus $f'(a)$ is invertible, and we may multiply through by its inverse to obtain $g'(b) = h'(a) \cdot (f'(a))^{-1} = (f'(a))^{-1}$. □

(3a) Let $R = [a, b] \times [c, d]$ be a closed, bounded rectangle in $\mathbb{R}^2$. Suppose that f is a continuously differentiable function defined in some open set containing R, and that $D_{2,2} f(x, y) > 0$ for all $(x, y) \in R$. Let $z_0 = (x_0, y_0)$ be some point in $(a, b) \times (c, d)$ with the following properties: (i) $f(x_0, y_0) = \min_{y \in [c,d]} f(x_0, y)$, and (ii) $f(z_0) = \max_{x \in [a,b]} \left(\min_{y \in [c,d]} f(x, y) \right)$. Show that $f'(z_0) = 0$.

Solution: Consider the function $h(x, y) = D_2 f(x, y)$. Since $D_2 h(z) \neq 0$, we may apply the implicit function theorem to conclude that there exist $\delta, \tilde{\delta} > 0$ such that whenever $|x - x_0| < \delta$, there exists a unique point $c(x) \in (y_0 - \tilde{\delta}, y_0 + \tilde{\delta})$ satisfying $h(x, c(x)) = 0$. Moreover the function c is differentiable.

For any $x \in [a, b]$, the function $g_x(y)$ satisfies $g_x''(y) = D_{2,2} f(x, y)$, so g_x has a strictly positive second derivative. Therefore by one variable calculus we know that $\min_{y \in [c,d]} f(x, y) = f(x, c(x))$ whenever $|x - x_0| < \delta$, provided that $\delta > 0$ is sufficiently small.

Now consider the function $\varphi(x) = f(x, c(x))$. The hypothesis that $f(z_0)$ equals $\max_{x \in [a,b]} \left(\min_{y \in [c,d]} f(x, y) \right)$ tells us that φ has a local maximum at $x = x_0$.

By the chain rule, φ is a differentiable function in $(x_0 - \delta, x_0 + \delta)$, and $\varphi'(x) = D_1 f(x, c(x)) + D_2 f(x, c(x)) \cdot c'(x)$. Since φ has a local maximum at x_0, $\varphi'(x_0) = 0$. Recall that $c(x_0) = y_0$. We already know that $D_2 f(z) = D_2 f(x_0, c(x_0)) = 0$, so $0 = \varphi'(x_0) = D_1 f(x_0, c(x_0)) = D_1 f(z)$. Since both partial derivatives of f vanish at z, we conclude that $f'(z) = 0$ by the theorem relating the Jacobian matrix of a differentiable function to its partial derivatives. □

(3b) Let $f, g : \mathbb{R}^2 \to \mathbb{R}^1$ be continuously differentiable functions, and let $a \in \mathbb{R}^2$. Suppose that $f(a) = 0$, and that $D_2 f(a) \neq 0$. Let $S = \{x \in \mathbb{R}^2 : f(x) = 0\}$. Suppose that there exists $\delta > 0$ such that for all $x \in S \cap B(a, r)$, $g(x) \leq g(a)$ (that is, on the set S, g has a local maximum at a). Show that there exists some real number t such that $g'(a) = tf'(a)$.

Solution: By the implicit function theorem, there exist an open rectangle $R = (a^1 - \delta, a^1 + \delta) \times (a^2 - \varepsilon, a^2 + \varepsilon)$ containing $a = (a^1, a^2)$ and a differentiable function $h : (a^1 - \delta, a^1 + \delta) \to (a^2 - \varepsilon, a^2 + \varepsilon)$ such that $S \cap R = \{(x, h(x)) : |x - a^1| < \delta\}$.

The function $G(x) = g(x, h(x))$ has a local maximum at $x = a^1$, by hypothesis. Moreover G is differentiable for all $x \in (a^1 - \delta, a^1 + \delta)$, provided that $\delta > 0$ is chosen to be sufficiently small, by the chain rule. Thus $G'(a^1) = 0$.

By the chain rule, for all $x \in \mathbb{R}^1$ satisfying $|x - a^1| < \delta$, $G'(x) = D_1 g(x, h(x)) + D_2 g(x, h(x)) \cdot h'(x)$. Thus the row vector $g'(a) = (D_1 g(a), D_2 g(a))$ is orthogonal to $(1, h'(a^1))$.

We also have $f(x, h(x)) = 0$ whenever $|x - a^1| < \delta$, so the derivative of this function with respect to x is also zero. But by the chain rule, that derivative is $D_1 f(x, h(x)) + D_2 f(x, h(x))h'(x)$. Thus the row vector $f'(a) = (D_1 f(a), D_2 f(a))$ is also orthogonal to the vector $(1, h'(a^1))$. Since $f'(a), g'(a)$ are orthogonal to the same nonzero vector in a two-dimensional vector space, they are colinear; since $f'(a) \neq 0$, $g'(a)$ is a scalar multiple of $f'(a)$. □

Comment. For the problem as originally written, the solution is essentially the same, but the notation is different. Adopt coordinates $(x, y) \in \mathbb{R}^n$ with $x \in \mathbb{R}^{n-1}, y \in \mathbb{R}^1$. Write $a = (\tilde{a}, a_n)$ in these coordinates. Now $G(x, h(x))$ is defined for all x in some open set in $\mathbb{R}^{n-1}$ containing $\tilde{a}$, and since G has a local maximum at $\tilde{a}$, we conclude that $D_i G(\tilde{a}) = 0$ for all $1 \leq i \leq n - 1$. By the chain rule, $D_i G(x) = D_i g(x, h(x)) + D_n g(x, h(x)) \cdot D_i h(x)$. This equals zero, for every $i \in \{1, 2, \cdots, n-1\}$. Likewise the same holds for the function $\mapsto f(x, h(x))$. Thus both $g'(a)$ and $f'(a)$ are orthogonal to $e_i + D_i h(a) e_n$ for each $i \in \{1, 2, \cdots, n - 1\}$. These vectors are clearly linearly independent, so they span an $n-1$-dimensional subspace of $\mathbb{R}^n$. Their mutual orthocomplement is a one-dimensional subspace, to which both $g'(a), f'(a)$ belong. Since $f'(a) \neq 0$, this forces $g'(a)$ to be a scalar multiple of $f'(a)$. □

Please begin reading the text of Stroock. Read §2.0 for Friday, and then §2.1 for next week.