Mathematics 105, Spring 2004 — Midterm Exam #1 Comments

(1b) Give an example of a function $f : \mathbb{R}^3 \to \mathbb{R}^1$ such that f is not differentiable at $a = (0, 0, 0)$, but all partial derivatives $D_i f(a)$ do exist.

Comment: A common answer was $f(x, y, z) = xyz/\sqrt{x^2 + y^2 + z^2}$ if $(x, y, z) \neq 0$, and $f(0, 0, 0) = 0$. This doesn't work. Indeed, $|f(x, y, z)| \leq |(x, y, z)|^2$, and hence $f'(0, 0, 0)$ equals the zero matrix (by problem (1c)).

Some students wrote formulas like $xyz/\sqrt{x^2 + y^2 + z^2}$ without addressing the issue of division by zero. It's necessary to explicitly define $f(0, 0, 0)$.

Several used $f(x, y, z) = xy/\sqrt{x^2 + y^2}$, and attempted to deal with the issue of division by zero by saying that the formula defines f when $(x, y) \neq (0, 0)$, and defined $f(x, y, z) = 0$ whenever $(x, y) \neq (0, 0)$. This doesn't handle the problem of division by zero; the denominator $\sqrt{x^2 + y^2}$ vanishes at every point of the form $(0, 0, z)$, not only at $(0, 0, 0)$. □

(3a) Let $R = [a, b] \times [c, d]$ be a closed, bounded rectangle in $\mathbb{R}^2$. Suppose that f is a continuously differentiable function defined in some open set containing R, and that $D_{2,2} f(x, y) > 0$ for all $(x, y) \in R$. Let $z_0 = (x_0, y_0)$ be some point in $(a, b) \times (c, d)$ with the following properties: (i) $f(x_0, y_0) = \min_{y \in [c,d]} f(x_0, y)$, and (ii) $f(z_0) = \max_{x \in [a,b]} \left(\min_{y \in [c,d]} f(x, y)\right)$. Show that $f'(z_0) = 0$.

Comment: A very common error was to assert that the function $\varphi(x) = f(x, y_0)$ has a local maximum at $x = x_0$, and hence $D_1 f(x_0, y_0) = \varphi'(x_0) = 0$. This is not true. Consider the example $f(x, y) = (y - 2x)^2 - x^2$ with $(x_0, y_0) = (0, 0)$. Set $[a, b] = [-1, 1]$ and $[c, d] = [-4, 4]$. Then $\min_{|y| \leq 4} f(x, y)$ is attained when $y = 2x$. Since $f(x, 2x) = 0^2 - x^2 = -x^2$, the "maximin" $\max_{|x| \leq 1} \min_{|y| \leq 4} f(x, y)$ is attained at (x_0, y_0) (and at no other point).

On the other hand, $f(x, y_0) = f(x, 0) = 3x^2$. This function does <u>not</u> have a local maximum at $x = x_0 = 0$; indeed, it has a strict local minimum there!

Notice that the conclusion of the problem is nonetheless correct: $f'(0, 0) = (0, 0)$. □

Comment: I gave you a complete solution of this problem in the solution to homework problem 2-41 from Spivak.

(3b) Let $f, g : \mathbb{R}^2 \to \mathbb{R}^1$ be continuously differentiable functions, and let $a \in \mathbb{R}^2$. Suppose that $f(a) = 0$, and that $D_2 f(a) \neq 0$. Let $S = \{x \in \mathbb{R}^2 : f(x) = 0\}$. Suppose that there exists $r > 0$ such that for all $x \in S \cap B(a, r)$, $g(x) \leq g(a)$ (that is, on the set S, g has a local maximum at a). Show that there exists some real number t such that $g'(a) = t f'(a)$.

Comment: A common error was to assert that g has a local maximum at a. This is untrue; consider the example $f(x, y) = y$, $g(x, y) = y - x^2$, and $a = (0, 0) \in \mathbb{R}^2$. Then S is the horizontal axis $\{(x, y) : y = 0\}$, and since $g(x, 0) = -x^2$, we do have $g(z) \leq g(a)$ for all $z \in S$.

On the other hand $g = y - x^2$ clearly has no local maximum at a when it is regarded as a function from (an open subset of) $\mathbb{R}^2$ to $\mathbb{R}$. $\square$

Comment: A common problem is to maximimize or minimize a function g, subject to a constraint (here $f(x, y) = 0$). We've derived (in the special case where $n = 2$ and there is a single "one-dimensional" constraint) a necessary condition which relates the derivative of g to the constraint. It's not a sufficient condition, only a necessary one (just as the condition $g'(x) = 0$ is only a necessary condition in one variable calculus). The parameter t is called a Lagrange multiplier.

The general statement is this: Assume that g is real-valued and defined in an open subset V of $\mathbb{R}^n$. Let $S = \{x \in V : f_k(x) = 0 \text{ for all } 1 \leq k \leq r\}$ where $r < n$ and each f_k is a real-valued function from V to $\mathbb{R}^1$. Suppose that $a \in S$, that the vectors $f_1'(a), \cdots, f_r'(a)$ are linearly independent, and that g, f_k are all continuously differentiable in V. If the restriction of g to S has a local maximum at a, then the vector $g'(a)$ belongs to the span of $\{f_1'(a), \cdots, f_r'(a)\}$. (Here we regard the $n \times 1$ Jacobian matrices as vectors.) $\square$

Distribution of scores: There were 50 points possible. Scores ranged from 41 to 17, with median 31. Most scores were near the median; out of 23 students, 13 scored between 26 and 34, 5 scored from 38 to 41, and 4 scored 17-18 with one 23. If letter grades were given based on this exam, 38-41 would be an A, 26-34 would be the B to A minus range, and 17-18 would be a C.

Final course grades will be based 15% on this exam, 20% on the second midterm exam, 20% on problem sets, and 45% on the final exam.