Mathematics 105, Spring 2004 — M. Christ
Final Exam Solutions (selecta) [1]

(2e) True or false: If $K \subset \mathbb{R}$ is a compact set of Lebesgue measure zero, and if $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism (that is, f is continous and invertible, and f^{-1} is also continuous), then $|f(K)| = 0$.

Solution. Like all the true/false questions, this is false. Take K to be the Cantor set $\mathcal{C} \subset [0,1]$ which is defined by successively deleting middle thirds of intervals. $\mathcal{C} = \cap_{n=0}^{\infty} \mathcal{C}_n$ where $\mathcal{C}_n$ is a disjoint union of 2^n (particular) intervals of length 3^{-n}. Consider the piecewise linear function f_n which is constant on each interval in the complement of $\mathcal{C}_n$, and has constant slope equal to $3^n/2^n$ on each of the intervals comprising $\mathcal{C}_n$. It can be shown that the sequence f_n converges uniformly as $n \to \infty$ to a limiting function f, which is necessarily continuous.

Consider the function $g(x) = f(x) + x$. Then g is strictly increasing and is continuous, so is a bijection from $\mathbb{R}$ to $\mathbb{R}$, and also has a continuous inverse. Since $g(0) = 0$ and $g(1) = 2$, g maps $[0,1]$ bijectively to $[0,2]$. $[0,1] \setminus \mathcal{C}$ is a countable disjoint union of intervals I, the sum of whose lengths equals $|[0,1] \setminus \mathcal{C}| = 1 - 0 = 1$. On each of those intervals I, $g(x) = x$ plus a constant, so each I is mapped to an interval of length $|I|$. Their images are disjoint, so $|g([0,1] \setminus \mathcal{C})| = \sum_I |I| = 1$. Since $g(\mathcal{C}) = [0,2] \setminus \big(g([0,1] \setminus \mathcal{C})\big)$, it follows that $|g(\mathcal{C})| = 2 - 1 = 1 > 0$. $\square$

(4a) Let $(E, \mathcal{A}, \mu)$ be a measure space and for each $j = 1, 2, \cdots$ let $A_j \in \mathcal{A}$. Suppose that $\sum_{j=1}^{\infty} \mu(A_j)$ is finite. Let B be the set of all points $x \in E$ which belong to infinitely many of the sets A_j. Show that $\mu(B) = 0$.

Solution. Set $f_j = \chi_{A_j}$, $F_N = \sum_{j=1}^{N} f_j$, and $f = \lim_{N\to\infty} F_N = \sum_j f_j$. Since each f_j is nonnegative, $f(x)$ is well-defined for each $x \in E$ as an element of $[0, +\infty]$. Each partial sum F_N is a finite sum of measurable functions and hence is measurable (since $+\infty - \infty$ never arises in the formation of this sum). Since the limit of a sequence of measurable functions which converges (in the extended reals) at every point is again a measurable function, f is measurable.

Now $f \in L^1$, since

$$\int_E f\,d\mu = \int_E \lim_{N\to\infty} F_N\,d\mu = \lim_{N\to\infty} \int_E F_N\,d\mu = \lim_{N\to\infty} \int_E \sum_{j=1}^{N} f_j\,d\mu$$

$$= \lim_{N\to\infty} \sum_{j=1}^{N} \int_E f_j\,d\mu = \lim_{N\to\infty} \sum_{j=1}^{N} |A_j| = \sum_{j=1}^{\infty} |A_j| < \infty.$$

The second equality is justified by the Monotone Convergence Theorem, and the fourth by the fact that $\int g + h = \int g + \int h$ for any two integrable functions.

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[1]

Now consider $A = \{x : f(x) = +\infty\}$. This is a measurable set, by one of our homework problems. $\mu(A)$ must equal zero, since $\infty\chi_A \le f$ and therefore $\infty\mu(A) \le \int f\,d\mu < \infty$.

For any $x \in E$, $x \in A_j$ for infinitely many indices j if and only if $f(x) = \infty$. Therefore $B = A$. □

(4b) Let $(E, \mathcal{A}, \mu)$ be a measure space satisfying $\mu(E) = 1$. Let $f, g : E \to \mathbb{R}$ be bounded measurable functions. Suppose that for any Borel sets $A, B \subset \mathbb{R}$,

$$\mu(f^{-1}(A) \cap g^{-1}(B)) = \mu(f^{-1}(A)) \cdot \mu(g^{-1}(B)).$$

Prove that

$$\int_E fg\,d\mu = \int_E f\,d\mu \cdot \int_E g\,d\mu.$$

Solution.[2] First suppose that f, g are two simple functions which satisfy the hypotheses. Let $\{a_j\}$ be the range of f (that is, the finite set of values taken on by f; no two numbers a_j are equal), and let $A_j = f^{-1}(\{a_j\}) = \{x : f(x) = a_j\}$. Likewise let $\{b_k\}$ be the range of g, and set $B_k = \{x : g(x) = b_k\}$.

Now $A_j \cap B_k = f^{-1}(\{a_j\}) \cap g^{-1}(\{b_k\})$, so by hypothesis $\mu(A_j \cap B_k) = \mu(A_j)\mu(B_k)$. Therefore $\int \chi_{A_j}\chi_{B_k}\,d\mu = \mu(A_j)\mu(B_k)$.

Consequently

$$\begin{aligned}\int fg\,d\mu &= \int \sum_j a_j\chi_{A_j} \sum_k b_k\chi_{B_k}\,d\mu = \sum_{j,k} a_jb_k \int \chi_{A_j}\chi_{B_k}\,d\mu \\ &= \sum_{j,k} a_jb_k\mu(A_j)\mu(B_k) = \sum_j a_j\mu(A_j) \cdot \sum_k b_k\mu(B_k) = \int f\,d\mu \cdot \int g\,d\mu.\end{aligned}$$

□

Now for the general case. It doesn't follow from the case of simple functions just by taking limits. For if we approximate f, g by simple functions in a straightforward way, then those approximants won't (absent a stroke of incredible luck) satisfy the hypothesis.

Let $\varepsilon > 0$ be given. Fix a positive number M sufficiently large that $|f(x)| < M$ and $|g(x)| < M$ for all $x \in E$. Suppose we can find simple functions φ, ψ such that $|f(x) - \varphi(x)| < \varepsilon$ and $|g(x) - \psi(x)| < \varepsilon$ for all $x \in E$. Then

$$|\int fg\,d\mu - \int \varphi\psi\,d\mu| \le \int |fg - \varphi\psi|\,d\mu \le \int \big(|f-\varphi||g| + |\varphi||g-\psi|\big)\,d\mu$$

$$\le M\int |f-\varphi|\,d\mu + (M+\varepsilon)\int |g-\psi|\,d\mu \le M\varepsilon\mu(E) + (M+\varepsilon)\varepsilon\mu(E) = (2M+\varepsilon)\varepsilon.$$

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[2] If you're paying close attention you'll notice that the hypothesis that $\mu(E) = 1$ is never used. However, if we look at the hypothesis in the special case $A = B = \mathbb{R}$ and note that $f^{-1}(\mathbb{R}) = g^{-1}(\mathbb{R}) = E$, then we conclude that $\mu(E) = \mu(E)^2$, which forces $\mu(E)$ to be either 1 or 0.

I used the upper bounds $|g(x)| \le M$ and $|\varphi(x)| \le |f(x)| + \varepsilon \le M + \varepsilon$. Likewise $|\int f\,d\mu - \int \varphi\,d\mu| < \varepsilon\mu(E) = \varepsilon$ and $|\int g\,d\mu - \int \psi\,d\mu| < \varepsilon\mu(E) = \varepsilon$. Since $\int \varphi\psi = \int \varphi \cdot \int \psi$, we may combine these inequalities:

$$\left|\int fg - \int f \cdot \int g\right| \le \left|\int fg - \int \varphi\psi\right| + \left|\int \varphi\psi - \int \varphi \cdot \int \psi\right| + \left|\int \varphi \int \psi - \int f \cdot \int g\right|.$$

The middle term on the right-hand side is zero, provided that φ, ψ satisfy the hypothesis. Thus $\left|\int fg - \int f \cdot \int g\right| \le (2M+\varepsilon)\varepsilon + 2\varepsilon$. Therefore the proof would be complete, if for any $\varepsilon > 0$ we could construct such approximants φ, ψ.

To do this, let $\varepsilon > 0$ be given, and set $A_j = \{x : f(x) \in [-M+j\varepsilon, -M+(j+1)\varepsilon)\}$. Then define $a_j = -M + j\varepsilon$, and $\varphi = \sum_j a_j \chi_{A_j}$. Similarly set $B_k = \{x : g(x) \in [-M+k\varepsilon, -M+(k+1)\varepsilon)\}$, $b_k = -M + k\varepsilon$, and $\psi = \sum_k b_k \chi_{B_k}$. Note that only finitely many values of j, k are relevant, since A_j is empty whenever j is sufficiently large that $-M + j\varepsilon > M$, and B_k is empty whenever k is sufficiently large that $-M + k\varepsilon > M$. By restricting to those indices j, k we obtain finite sums. φ, ψ are thus simple functions, and from their definitions it is apparent that $|f(x) - \varphi(x)| < \varepsilon$ for all x and likewise for $|g - \psi|$.

In order for the above proof for the case of simple functions to be applicable, we merely need to know that $\mu(A_j \cap B_k) = \mu(A_j) \cdot \mu(B_k)$ for all j, k. But $A_j = f^{-1}([-M+j\varepsilon, -M+(j+1)\varepsilon))$ and likewise B_k equals the inverse image under g of a certain set, so this follows directly from the hypothesis for f, g. □

(4c) Let $A \subset \mathbb{R}^2$ be a Borel set. Suppose that for each $c \in \mathbb{R}$, the set $\{(x_1, x_2) \in A : x_1 - x_2 = c\}$ is a finite set. Prove that the Lebesgue measure $|A|$ equals zero.

Solution. Define an invertible linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x_1, x_2) = (x_1, x_2 - x_1)$. Set $B = T(A)$. By hypothesis, B has the property that for any $c \in \mathbb{R}$, there are only finitely many points in B whose second coordinate equals c; that is, the intersection of B with any horizontal line is finite.

Since T has determinant equal to one, so does T^{-1}. Since A is a Borel set and T^{-1} is a continuous function, B is also a Borel set, and $|B| = |A|$. Thus it suffices to prove that $B| = 0$.

We have noted that $\mathcal{B}_{\mathbb{R}^1} \times \mathcal{B}_{\mathbb{R}^1} = \mathcal{B}_{\mathbb{R}^2}$, so we may apply Tonelli's theorem to χ_B:

$$|B| = \int_{\mathbb{R}^2} \chi_B \, d\lambda_{\mathbb{R}^2} = \int_{\mathbb{R}^1} \int_{\mathbb{R}^1} \chi_B(x_1, x_2)\, d\lambda(x_1)\, d\lambda(x_2),$$

where λ denotes Lebesgue measure on $\mathbb{R}^1$. The inner integral $\int_{\mathbb{R}^1} \chi_B(x_1, x_2)\, d\lambda(x_1)$ is the Lebesgue measure of $\{x_1 : (x_1, x_2) \in B\}$. As remarked above, this set is finite, hence has measure zero. Therefore the inner integral is zero for every $x_2 \in \mathbb{R}$, and hence

$$|B| = \int_{\mathbb{R}^1} 0 \, d\lambda = 0.$$

□