Mathematics 105, Spring 2004 — M. Christ
Final Exam Review Guide

The final exam will primarily emphasize the portion of the course concerned with Lebesgue integration, in which we followed Stroock's Chapters 2, 3, §4.1 and §§5.0,2,3. From the first part of the course, you should know definitions, statements of theorems including hypotheses, and examples. You might be asked to state or to apply theorems, but you will not need to know details of proofs or of solutions to the problem sets.

In format the final exam will be similar to the midterms. There will be short answer questions concerning definitions and examples; some of the latter may be in True-False format as in the second midterm exam. There will also be (short answer) questions about statements of theorems. Longer questions may ask you to reproduce portions of proofs of theorems and lemmas, to reproduce solutions of homework problems, or to solve new problems.

I will aim for this exam to be approximately twice as long as the midterm exams; you'll have three times as much time.

Definitions. Compact sets. Open and closed sets. Differentiable functions, $Df(a)$. Partial derivatives, Jacobian matrix. Directional derivatives, and their connection with Df. Second-order partial derivatives, continuously differentiable functions.

σ-algebra, measure, measure space, measurable space. Probability measures, finite measures, σ-finite measures. Outer measure, Lebesgue measurable sets, Lebesgue measure. $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$ sets. Countable additivity, countable subadditivity. Complete measures, completion of a measure space. Borel sets, the Borel σ-algebra associated to any metric space. The σ-algebra $\sigma(E;\mathcal{C})$ generated by a collection $\mathcal{C}$ of subsets of E. π-systems and λ-systems. Axiom of choice.

Characteristic (or indicator) functions, simple functions. Measurable functions, alternative characterizations of measurability. Arithmetic in the extended real numbers. $\int f\,d\mu$ (for various classes of functions). Integrable functions, L^1. Null sets, equality almost everywhere, the equivalence relation $f \sim g$ which makes L^1 into a normed linear space, and in particular into a metric space.

$\liminf$ and $\limsup$ of a sequence of (extended real) numbers. (You should know the characterization of $\limsup_{n\to\infty} a_n$ as $\lim_{k\to\infty} \sup_{n\ge k} a_n$, and the corresponding characterization of $\liminf a_n$). Almost everywhere convergence. Convergence in measure. A sequence of functions is Cauchy in measure, or Cauchy in L^1. Convergence in L^1 norm.

Complete metric space. μ is absolutely continuous with respect to ν (see Exercise 3.3.15).

Hardy-Littlewood maximal function, Hardy-Littlewood maximal inequality, the notation $I \searrow \{x\}$.

Product of two σ-algebras, product of two measures.

$\Phi_*(\mu)$. S^{n-1}, $\lambda_{S^{n-1}}$. Rotation-invariant measures.

Theorems, et cetera. (The following are not precise statements.)

(†) A subset of $\mathbb{R}^n$ is closed and bounded if and only if it is compact; compactness (defined in terms of open covers) is equivalent to every sequence having a convergent subsequence.

(†) Uniqueness of the derivative. Chain rule. Product and sum rules. Connection between local maxima/minima and vanishing of derivative. Connection between derivative and partial derivatives. Equality of mixed second partial derivatives (proved near end of course).

(†) Inverse function theorem. Implicit function theorem (two versions). In these results it is particularly important to understand the hypotheses.

(†) Lebesgue measure exists, and has certain properties . . . Every subset of a null set is measurable. Every Lebesgue measurable set is sandwiched between an $\mathcal{F}_\sigma$ and a $\mathcal{G}_\delta$.

(†) Measure and measurability of $T(A)$, where T is a linear transformation.

(†) Connection between π-systems, λ-systems, and σ-algebras (Lemma 3.1.3).

(†) Measure of limit of an ascending, or descending, sequence of sets. Integral of limit of an ascending sequence of simple functions (Lemma 3.2.6). (This lemma was central to the whole theory; the proof of the Monotone Convergence Theorem relied mainly on this lemma, and in the last weeks of the course we repeatedly used Monotone Convergence to prove other things.)

(†) Measurability of sums and products of measurable functions. Integral of a sum. Markov's inequality. The sum of two L^1 functions belongs to L^1. L^1 is a normed linear space.

(†) If $f : [a, b] \to \mathbb{R}$ is continuous, then its Riemann integral equals its Lebesgue integral. (We showed this in a homework problem.)

(†) The Monotone and Dominated Convergence Theorems, and Fatou's Lemma. (We did not cover the slightly fancier versions in Theorem 3.3.12.)

(†) Convergence in L^1 norm implies convergence in measure. If a sequence of functions is Cauchy in measure it has a subsequence which converges almost everywhere. Any Cauchy sequence in L^1 converges to a limit in L^1, in L^1 norm. (Thus $L^1(E, \mathcal{A}, \mu)$ is always a complete metric space.)

(†) If $f \in L^1(\mu)$ then $\nu(A) = \int_A f\, d\mu$ defines a measure. This measure is absolutely continuous with respect to μ.

(†) For any metric space and any Borel measure satisfying a certain technical hypothesis similar to σ-finiteness, the set of all (uniformly) continuous functions is dense in L^1.

(†) Lebesgue's differentiation theorem. Statement (but not the proof) of the Hardy-Littlewood maximal inequality.

(†) Fubini's and Tonelli's theorems. (You are not responsible for the notion of a semilattice; recall that in class we structured the proofs of these theorems so as to avoid this notion.) Versions of Fubini's and Tonelli's theorems for the completion of $(E_1 \times E_1, \mathcal{A}_1 \times \mathcal{A}_2, \mu_1 \times \mu_2)$, when $(E_j, \mathcal{A}_j, \mu_j)$ are themselves complete.

(†) The theorem on integration in polar coordinates, relating $R_n \times \lambda_{S^{n-1}}$ to $\lambda_{\mathbb{R}^n}$. Jacobi's transformation formula for changes of variables in Lebesgue integrals.

Examples. $\star$ A nonmeasurable (with respect to $\overline{\mathcal{B}_\mathbb{R}}$) subset of $\mathbb{R}$. $\star$ The Cantor ternary set (that's the one obtained by deleting middle thirds of intervals), and "fat" or "generalized" Cantor sets. $\star$ The Cantor ternary functions. These are a continuous function which maps the Cantor ternary set onto $[0,1]$, and a continuous strictly increasing function which maps the Cantor ternary set onto a subset of $[0,2]$ which has positive Lebesgue measure. $\star$ An open dense subset of $\mathbb{R}$ with arbitrarily small measure. $\star$ A sequence of functions which converge to zero in measure, but not almost everywhere. $\star$ Various examples relevant to the theorems on convergence of integrals; these showed that the hypotheses in those theorems cannot be dispensed with. $\star$ A null set in $\mathbb{R}^2$ which does not belong to $\overline{\mathcal{B}}_{\mathbb{R}^1} \times \overline{\mathcal{B}}_{\mathbb{R}^1}$. $\star$ A function defined on $[0,1]$ which is Lebesgue integrable but is not Riemann integrable.

Some proofs to know. Here are a few (mostly) basic facts whose proofs I'd particularly like you to know. Two of these will be on the exam, essentially verbatim. (You are of course responsible for all proofs discussed in the portion of the course concerning Lebesgue integration.)

1. If μ is a measure defined on $\mathcal{B}_{\mathbb{R}^n}$, if μ is translation-invariant and if $\mu(Q) = |Q|$ for every (closed) cube $Q \subset \mathbb{R}^n$, then $\mu(A) = |A|$ for every set $A \in \mathcal{B}(\mathbb{R}^n)$.

2. Let $\{A_j\}$ be measurable sets in some measure space. Show that if $A_j \supset A_{j+1}$ for all j then $\mu(A_j) \to \mu(\cap_k A_k)$ as $j \to \infty$, provided that there exists m such that $\mu(A_m) < \infty$. Show that if instead $A_j \subset A_{j+1}$ for all j then $\mu(A_j) \to \mu(\cup_k A_k)$.

3. If $G \subset \mathbb{R}^n$ is open, and if $f : G \to \mathbb{R}^n$ is Lipschitz continuous (that is, there exists $C < \infty$ such that $|f(x) - f(y)| \le C|x-y|$ for all $x, y \in G$), then for any Lebesgue measurable set $A \subset G$, $f(A)$ is Lebesgue measurable.

4. If f, g are real-valued measurable functions, then $f + g$ is likewise measurable.

5. Let $(E, \mathcal{A}, \mu)$ be a measure space satisfying $\mu(E) < \infty$. If φ, φ_n are nonnegative simple measurable functions, if $\varphi_n \le \varphi_{n+1}$, and if $\varphi(x) = \lim_{n\to\infty} \varphi_n(x)$ for all $x \in E$, then $\int \varphi_n \, d\mu \to \int \varphi \, d\mu$ as $n \to \infty$.

6. Lebesgue's Dominated Convergence Theorem.

7. If $f \in L^1(\mu)$ then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta \Rightarrow \int_A |f| \, d\mu < \varepsilon$.

8. Let μ_j be measures on $\mathcal{A}_j$ for $j = 1, 2$. Suppose that $\mu(S) = \nu(S)$ whenever $S = A_1 \times A_2$ and $A_j \in \mathcal{A}_j$. Show that $\mu(S) = \nu(S)$ for every set $S \in \mathcal{A}_1 \times \mathcal{A}_2$. Explain the key role played by this result in the proof of Tonelli's theorem.

9. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous, and for $r \in (0, \infty)$ define $F(r) = \int_{|x| \le r} f(x) \, d\lambda_{\mathbb{R}^n}(x)$. Show that $F'(r) = r^{n-1} \int_{S^{n-1}} f(r\omega) \, d\lambda_{S^{n-1}}(\omega)$.

Besides the text and lecture notes, don't forget to study the problem sets and their solutions, and problems from the two midterm exams.