The universality of polynomial time Turing equivalence

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Abstract

We show that polynomial time Turing equivalence and a large class of other equivalence relations from computational complexity theory are universal countable Borel equivalence relations. We then discuss ultrafilters on the invariant Borel sets of these equivalence relations which are related to Martin's ultrafilter on the Turing degrees.

1 Introduction

1.1 Universal resource bounded equivalence relations from computability

In this paper, we study the global complexity of resource bounded reducibilities from computability theory. If $\mathcal{C}$ is a subset of the Turing reductions such as the polynomial time Turing reductions, and $x, y \in \mathcal{P}(2^{<\omega})$ are languages, then we write $x \leq_T^{\mathcal{C}} y$ if x is Turing reducible to y via a Turing reduction in $\mathcal{C}$. Similarly, if $\mathcal{C}$ is a subset of the many-one reductions, then we analogously write $x \leq_m^{\mathcal{C}} y$ if x is many-one reducible to y via a many-one reduction from $\mathcal{C}$. Now these relations are not transitive for arbitrary $\mathcal{C}$. However, we will be particularly interested in the special case where we fix a function $g\colon \mathbb{N} \to \mathbb{N}$ which is strictly increasing, time constructible, and $g(n) \geq n^2$, and let $\mathcal{C}$ be the union over k of all reductions computable in $O(g^k)$-time or $O(g^k)$-space. Note that here we use g^k to denote the k-fold

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composition of g, $g^k = \underbrace{g \circ \ldots \circ g}_{\text{k times}}$, and not its k-fold product. In this case, since there is only a polynomial amount of overhead required to simulate the composition of two reductions, it is easy to see that the associated reducibilities are transitive and symmetrize to equivalence relations which we study below.

Previous investigations of the global properties of resource bounded reducibilities in computability have focused mainly on the theory of these structures [1,6,10,17]. We take a different approach, using the framework of Borel reducibility between Borel equivalence relations. A *Borel equivalence relation* E on a standard Borel space X is an equivalence relation that is a Borel subset of $X \times X$. If E and F are both Borel equivalence relations on X and Y, then E is *Borel reducible* to F, noted $E \leq_B F$, if there is a Borel function $f\colon X \to Y$ such that for all $x, y \in X$, we have $x\ E\ y \iff f(x)\ F\ f(y)$. Such a function induces an injection $\hat{f}\colon X/E \to Y/F$, and we can view Borel reducibility as comparing the difficulty of classifying E and F by invariants, where if $E \leq_B F$, then any complete set of invariants for F can be used as a set of complete invariants for E. Borel reducibility is also sometimes viewed as describing *Borel cardinality*, where the injection $\hat{f}$ is a Borel witness that the quotient of E injects into the quotient of F.

A Borel equivalence relation is said to be *countable* if all of its equivalence classes are countable. See [8] for an introduction to the theory of countable Borel equivalence relations. It is an important fact that there are maximal countable Borel equivalence relations under $\leq_B$, and such equivalence relations are said to be *universal* [5]. Our first theorem is that the resource bounded equivalence relations discussed above are universal (see Theorem 3.1).

Theorem 1.1. *Suppose $g(n) \geq n^2$ is a strictly increasing time-constructible function. Then any countable Borel equivalence relation E such that $\equiv_m^{\mathbf{TIME}(\cup_k O(g^k))} \subseteq E \subseteq \equiv_T^{\mathbf{SPACE}(\cup_k O(g^k))}$ is a universal countable Borel equivalence relation.*

For example, polynomial time many-one equivalence and polynomial time Turing equivalence are both universal countable Borel equivalence relations, corresponding to the case $g(n) = n^2$. It is an open question of Kechris [9] whether Turing equivalence is a universal countable Borel equivalence relation.

1.2 Ultrafilters, universality, and relativization

Suppose E is a countable Borel equivalence relation on a standard Borel space X. We say that a set $A \subseteq X$ is *E-invariant* if $\forall x, y \in X$ if xEy, then $x \in A \iff y \in A$. Of course, the Borel E-invariant sets form a σ-algebra. Ultrafilters on this σ-algebra of invariant Borel sets are important tools used in the study of Borel reducibility of equivalence relations. Such ultrafilters arise naturally from E-ergodic probability measures on X and E-generically ergodic Polish topologies on X. More recently, work started by Simon Thomas in [20] and continued in [12] and [11] has used ultrafilters related to Martin's ultrafilter on the Turing degrees to derive structural consequences about universal equivalence relations of various kinds. For example, in [12], these ultrafilters from computability were used to show that if E is a universal countable Borel equivalence relation on a standard Borel space X, and $A \subseteq X$ is Borel, then either $E \restriction A$ is universal or $E \restriction (X \setminus A)$ is universal. This answered a question of Jackson, Kechris, and Louveau [8]. These ultrafilters which preserve the Borel cardinality of their associated equivalence relations are promising candidates for resolving other questions from the theory of countable Borel equivalence relations known to be resistant to measure and category techniques. See [11] for an introduction to them.

It is an interesting problem to find other natural examples of such ultrafilters on countable Borel equivalence relations. For example, [21] has investigated whether there could be a cone measure for the quasiorder of embeddability among countable groups. Here, we consider such ultrafilters for resource bounded equivalence relations from computability theory.

The ultrafilters we consider are defined via games very similar to the game defining Martin's ultrafilter on the Borel Turing invariant sets (see [13]). Martin considered the game G_A where two players alternate defining longer and longer initial segments of an oracle x, and at the end, player I wins if the resulting oracle is in the payoff set A. Martin has shown [13] that player I has a winning strategy if and only if A contains a *Turing cone*, that is, a set of the form $\{x : x \geq_T y\}$ for some oracle y. Similarly, player II has a winning strategy if and only if the complement of A contains a Turing cone. Hence, by Borel determinacy [14], the set of Borel Turing-invariant A that contain a Turing cone forms an ultrafilter on the Borel Turing-invariant sets, which is known as *Martin's ultrafilter* or *Martin measure*.

It is folklore that Martin's proof generalizes to any quasi-order $\leq_Q$ such that given a strategy σ in the game and an oracle x which is $\geq_Q$ an encoding of σ, then $x \equiv_Q \sigma * x$ where $\sigma * x$ is the outcome of playing σ against x.

While this condition encompasses a large number of quasi-orders, it does not include Turing equivalence restricted to sub-exponential time or space bounds, since strategies in games take an exponential amount of space to encode. Nor does it include many-one equivalence, or its time or space restricted versions, since even if $x \geq_m \sigma$ where σ is a strategy in Martin's game, a many-one reduction relative to x cannot combine information about x and this strategy to determine $\sigma * x$. Here then, we may ask to what extent there can be an analogue of Martin's ultrafilter for these equivalence relations.

First, we note that standard relativization barriers show that there cannot be an ultrafilter defined via cones for these equivalence relations. For example, relativizing the theorem of Baker, Gill, and Solovay [2] giving oracles x and y relative to which $\mathbf{P}^x = \mathbf{NP}^x$ and $\mathbf{P}^y \neq \mathbf{NP}^y$ shows that both the oracles relative to which $\mathbf{P} = \mathbf{NP}$ and $\mathbf{P} \neq \mathbf{NP}$ are cofinal under polynomial time Turing reducibility $\leq_T^{\mathbf{P}}$, and so there is no cone ultrafilter on the $\equiv_T^{\mathbf{P}}$-invariant sets. Further, there can be no cone ultrafilter for $\leq_m$ since the collection of oracles that are many-one equivalent to their complement are cofinal under $\leq_m$ but do not contain a $\leq_m$-cone. However, it turns out that Martin's game still defines an ultrafilter with interesting structure-preserving properties, even though it does not have a definition via cones.

Theorem 1.2. *Let E be a countable Borel equivalence relation as in Theorem 1.1. Let G_A be Martin's game in* [13] *where the two players I and II alternate defining the elements of an oracle x and I wins if $x \in A$. Then the collection of E-invariant Borel sets A such that player I has a winning strategy in the game G_A is an ultrafilter on the Borel E-invariant sets. Furthermore, for any $A \in U$, $E \restriction A$ is a universal countable Borel equivalence relation.*

There is some connection here with issues in computability and complexity theory surrounding the phenomenon of relativization. In recursion theory, proofs almost always relativize. That is, we can take most any proof in the subject, change all Turing machines used in the proof to grant them access to an additional oracle x, and the proof will remain valid. And while early sweeping attempts to formalize this phenomenon failed, such as Rogers' homogeneity conjecture (see [15, §13.1], [16, §12], [18], and [19]), Martin's ultrafilter on the Turing degrees yields a somewhat weaker explanation of the ubiquity of this phenomenon. If $\psi(\mathbf{x})$ is a Borel property of a Turing degrees, then either $\psi(\mathbf{x})$ is true on a cone of oracles, or $\psi(\mathbf{x})$ is false on

a cone of oracles. Hence, every reasonable fact about the Turing degrees eventually relativizes above some Turing cone.

As we've noted above, Baker, Gill, and Solovay's result implies that the polynomial time analogue of this result is false. However, since Martin's cone theorem becomes true once we enlarge beyond iterated exponential time Turing equivalence, we can show that the connection between sets in our ultrafilter and cones (and hence the meta-theorem preventing the existence of relativization barriers on a cone) occurs once E grows beyond iterated exponential time Turing equivalence. That is, if E contains $\equiv_T^{\mathbf{ELEMENTARY}}$, then $A \in U$ if and only if A contains a $\leq_T^{\mathbf{ELEMENTARY}}$-cone. This fact tells us something interesting about relativization barriers themselves – that on a cone, such barriers can always be found inside the exponential hierarchy. Suppose ψ is a $\equiv_m^{\mathbf{P}}$ invariant property (such as whether two naturally defined relativized complexity classes coincide). Suppose also that there is a relativization barrier for ψ *which itself relativizes*, so that for every z, there are $x \geq_m^{\mathbf{P}} z$ and $y \geq_m^{\mathbf{P}} z$ relative to which $\psi(x)$ is true, and $\psi(y)$ is false. Then for a $\leq_m^{\mathbf{P}}$-cone of z, we can find such x and y so that $x, y \in \mathbf{ELEMENTARY}^z$. A consequence of this is that if ψ is invariant on $\equiv_T^{\mathbf{ELEMENTARY}}$-classes, then it must be either true or false on some $\leq_T^{\mathbf{ELEMENTARY}}$-cone, and hence ψ can not admit a Baker-Gill-Solovay-type relativization barrier.

In another direction, we show that the set of languages $x \in \mathcal{P}(2^{<\omega})$ for which $\mathbf{P}^x = \mathbf{NP}^x$ is in the ultrafilter U defined in Theorem 1.2. While oracles relative to which $\mathbf{P} = \mathbf{NP}$ are often considered to be somehow "rare" (for example meager [4] and Lebesgue null [3]) here we have a natural sense in which the set of oracles relative to which $\mathbf{P} = \mathbf{NP}$ is large–it has the maximal possible Borel cardinality.

2 Preliminaries

We begin by reviewing some notation and conventions. Given any set S, we will often exploit the bijection via characteristic functions between its powerset $\mathcal{P}(S)$, and 2^S, the space of functions from S to $\{0,1\}$, and move freely between these two representations. We use the notation 2^n for the set of finite binary strings of length n and $2^{\leq n}$ for finite binary strings of length $\leq n$. The set of all finite strings is noted $2^{<\omega}$. We use $r^\frown s$ to note the concatenation of the strings r and s.

Define $\mathcal{P}(2^{<\omega})$ to be the Polish space of subsets of $2^{<\omega}$. If $x, y \in \mathcal{P}(2^{<\omega})$, then their recursive join is $x \oplus y = \{0^\frown s : s \in x\} \cup \{1^\frown s : s \in y\}$. The recursive join of finitely many elements of $\mathcal{P}(2^{<\omega})$ is defined similarly. We use

the notation A^c to denote the complement of a set A. A function $g\colon \mathbb{N} \to \mathbb{N}$ is said to be *time constructible* if $g(n)$ is computable in $O(g(n))$-time.

In computational complexity, the issue of correctly relativizing and modeling oracle access is a delicate and complicated matter. See [7] for a discussion of some of these issues. The theorems we will prove, however, will be quite robust with respect to this issue, and the only assumption we will make is the standard convention that a computation which uses n space may query only oracle strings of length at most n.

If E and F are equivalence relations on the spaces X and Y, then a function $f\colon X \to Y$ is a *homomorphism from E to F* if for all $x, y \in X$ we have $x\ E\ y \implies f(x)\ F\ f(y)$. A function $f\colon X \to Y$ is a *cohomomorphism from E to F* if for all $x, y \in X$ we have $f(x)\ F\ f(y) \implies x\ E\ y$.

3 A universality proof

In this section, we prove the following theorem from the introduction.

Theorem 3.1. *Suppose $g(n) \geq n^2$ is a strictly increasing time-constructible function. Then any countable Borel equivalence relation E such that $\equiv_m^{\mathbf{TIME}(\cup_k O(g^k))} \subseteq E \subseteq \equiv_T^{\mathbf{SPACE}(\cup_k O(g^k))}$ is universal.*

Proof. We let E_∞ note a universal countable Borel equivalence relation that is generated by a continuous action of the free group on two generators $\mathbb{F}_2 = \langle \alpha, \beta \rangle$ on 2^ω. For example, we can use the shift action of $\mathbb{F}_2$ for the purpose (see [5]). We will show that E is universal by constructing a continuous reduction $\hat{f}$ from E_∞ to E.

The key to our proof is that given $\gamma \in \mathbb{F}_2$, we can code $\hat{f}(\gamma \cdot x)$ into $\hat{f}(x)$ rather sparsely so that if $|\gamma| = k$ is the length of γ as a reduced word, then strings of length n in $\hat{f}(\gamma \cdot x)$ are coded by strings of length greater than $g^k(n)$ in $\hat{f}(x)$. From here our basic idea is as follows: given $x, y \in 2^\omega$ such that $x \not E_\infty y$ and a Turing reduction that runs in g^k-time, we wait till we have finite initial segments of x and y witnessing that $\gamma \cdot x \neq y$ for any γ of length $\leq k$. Then if n is large enough, we can change the value of strings of length n in $\hat{f}(x)$ without changing $\hat{f}(y)$ restricted to strings of length $\leq g^k(n)$. This makes it easy to diagonalize. The remaining technical wrinkle of the proof is that we must be able to simultaneously do a lot of this sort of diagonalization. This proof is inspired by the proof of Theorem 2.5 in [12].

We now give a precise definition of the coding we will use. Let $c : 2^{<\omega} \to 2^{<\omega}$ be the function where $c(r) = 0^{g(|r|)} \frown 1 \frown r$ is $g(r)$ zeroes followed by a 1

followed by r. It is clear that if $x \in \mathcal{P}(2^{<\omega})$, then $c(x) = \{c(r) : r \in x\}$ is many-one reducible to x in $O(g)$-time, since g is time constructible.

Given $f : 2^\omega \to \mathcal{P}(2^{<\omega})$, let $\hat{f} : 2^\omega \to \mathcal{P}(2^{<\omega})$ be the unique function satisfying

$$\hat{f}(x) = f(x) \oplus c\left(\hat{f}(\alpha \cdot x) \oplus \hat{f}(\alpha^{-1} \cdot x) \oplus \hat{f}(\beta \cdot x) \oplus \hat{f}(\beta^{-1} \cdot x)\right).$$

While this definition of $\hat{f}$ is self-referential, note that if y is a language, and s is a string in y, then s is coded by a string of a greater length in $f(x) \oplus c(y)$. Hence, for each string s, there is a unique end-segment r_s of s and a word $\gamma_s \in \mathbb{F}_2$ so that s is in $\hat{f}(x)$ if and only if r_s is in $f(\gamma_s \cdot x)$. So $\hat{f}$ is uniquely defined.

Now it is clear that given any f, the associated $\hat{f}$ is a homomorphism from E_∞ to E, since $\equiv_m^{\mathbf{TIME}(\cup_k O(g^k))} \subseteq E$. We claim that if f is a sufficiently generic continuous function, then $\hat{f}$ will also be a cohomomorphism from E_∞ to $\equiv_T^{\mathbf{SPACE}(\cup_k O(g^k))}$ and hence also a cohomomorphism from E_∞ to E. By generic here, we mean for the following partial order for constructing a continuous (indeed, Lipschitz) function from 2^ω to $\mathcal{P}(2^{<\omega})$. Our partial order $\mathbb{P}$ will consist of functions $p : 2^n \to \mathcal{P}(2^{\leq n})$ such that if $m < n$ and $r_1, r_2 \in 2^n$ extend $r \in 2^m$, then $p(r_1)$ and $p(r_2)$ agree on all strings of length $\leq m$. Given a $p : 2^n \to \mathcal{P}(2^{\leq n})$ and $r \in 2^n$, we will often think of $p(r)$ as a function from $2^{\leq n}$ to 2. If p has domain 2^n, then we will say p has height n. If $p, p^* \in \mathbb{P}$ are such that the height of p is m and the height of p^* is n, and $m \leq n$, then say that p^* extends p, noted $p^* \leq_{\mathbb{P}} p$, if for all $r^* \in 2^n$ extending $r \in 2^m$, we have that $p^*(r^*)$ extends $p(r)$ (as functions).

If $p \in \mathbb{P}$, then we can define $\hat{p}$ analogously to the definition of $\hat{f}$ above. In particular, for each $\gamma \in \mathbb{F}_2$, we have some partial information about $p(\gamma \cdot r)$ based on the longest finite initial segment of $\gamma \cdot r$ that we know (recall that the action of $\mathbb{F}_2$ on 2^ω is continuous). Hence, given a finite string r, $\hat{p}$ will map r to a partial function from finite strings to 2 that amalgamates all this partial information.

Because of our coding scheme, if $r \in 2^{<\omega}$, the length of r is $|r| = n$, and $\gamma \in \mathbb{F}_2$ is of length k, then whether $r \in \hat{f}(\gamma \cdot x)$ is canonically coded into $\hat{f}(x)$ at some string of length greater than $g^k(n)$.

Let p_0 be the condition of height 1 where $p_0(r) = \emptyset$ for all r. Hence, if $f : 2^\omega \to \mathcal{P}(2^{<\omega})$ extends p_0, then every string in $f(x)$ must have length ≥ 2. For convenience, the generic function we construct will extend p_0. Note that since $g(n) \geq n^2$, if f is $O(g^{k-1})$ for some k, then $f(n) \leq g(n)^k$ for sufficiently large n.

Suppose we are given a Turing reduction φ_e that runs in g^k-space, and $r, s \in 2^m$ such that $\gamma \cdot r$ is incompatible with s for all $\gamma \in \mathbb{F}_2$ where $|\gamma| \leq k$. Let $D_{r,s,k,e}$ be the set of p of height $\geq m$ such that if p has height i, then there exists an $n \geq m$ so that if $t \in 2^i$, then $\hat{p}(t)$ is defined on all strings of length $\leq g^k(n)$, and for all $r^*, s^* \in 2^i$ extending r and s, we have that φ_e is not a g^k-time reduction of $\hat{p}(s^*)$ to $\hat{p}(r^*)$ as witnessed by some string t of length n, so that $\varphi_e(\hat{p}(r^*))(t) \neq \hat{p}(s^*)(t)$. We claim that $D_{r,s,k,e}$ is dense below p_0. The theorem will follow from this fact.

Given any $p \leq_{\mathbb{P}} p_0$ where p has height j, we must construct an extension p^* of p so that $p^* \in D_{r,s,k,e}$. Fix an n and an i such that $i \gg n \gg j$ with the precise bounds we give below. Define q of height i where for all r, $q(r)$ contains only the strings in $p(r \upharpoonright j)$. We require that $2^{j+1}4^{k+1}\log_2(n)^2 < n$, and that $i \gg n$ is large enough so that for all $r \in 2^i$, $\hat{q}(r)$ is defined on all strings of length $\leq g^k(n)$. Such an i exists by the continuity of the action of $\mathbb{F}_2$ on 2^ω.

Given any $r^* \in 2^i$ we compute an upper bound on how many elements $\hat{q}(r^*)$ could have of length $\leq g^k(n)$. Clearly every $q(t)$ has less than 2^{j+1} elements. It will be enough to establish an upper bound on the number of words $\gamma \in \mathbb{F}_2$ so that some element of $q(\gamma \cdot r^*)$ is coded into $\hat{q}(r)$ via a string of length $\leq g^k(n)$.

Since q extends p_0, any element of any $q(\gamma \cdot r^*)$ must be a string of length ≥ 2. Now if $\gamma \in \mathbb{F}_2$ is such that some string of length ≥ 2 in $q(\gamma \cdot r^*)$ is coded into $\hat{q}(r^*)$ below $g^k(n)$, then it must be that $g^{|\gamma|}(2) \leq g^k(n)$. Since $g(n) \geq n^2$, we have that $g^i(2) \geq 2^{2^i}$. Hence, $|\gamma| \leq k + \log_2\log_2(n)$. This is because otherwise

$$g^{|\gamma|}(2) \geq g^{k+\lceil \log_2 \log_2(n)\rceil}(2) = g^k \circ g^{\lceil \log_2 \log_2(n)\rceil}(2) \geq g^k(n)$$

and furthermore $g^{|\gamma|}(2) > g^{k+\lceil \log_2 \log_2(n)\rceil}$ if $\log_2\log_2(n)$ is an integer, and $g^{k+\lceil \log_2 \log_2(n)\rceil}(2) > g^k(n)$ if $\log_2\log_2(n)$ is not an integer, contradicting $g^{|\gamma|}(2) \leq g^k(n)$ either way. Thus, there are most $4^{k+1}\log_2(n)^2$ such group elements γ since there are at most 4^{l+1} words of $\mathbb{F}_2$ of length $\leq l$.

Since each $q(\gamma \cdot r^*)$ must have less than 2^{j+1} elements, $\hat{q}(r^*)$ contains at most $2^{j+1}4^{k+1}\log_2(n)^2$ strings of length $\leq g^k(n)$. Let S be this set of all possible strings in $\hat{q}(r^*)$ of length $\leq g^k(n)$, so we have picked n so that $|S| < n$. Note that S does not depend on r^*.

We see now that amongst all of the r^* extending r, there are $\leq 2^{n-1}$ possibilities for what any $\hat{q}(r^*)$ restricted to strings of length $\leq g^k(n)$ could be: they are all elements of $\mathcal{P}(S)$. Let $u_0, u_1, \ldots$ be a listing of the elements of $\mathcal{P}(S)$. Recall that based on our definition of recursive join and $\hat{p}$, strings

of length $n-1$ in $p(s^*)$ are coded into $\hat{p}(s^*)$ using strings of length n that begin with 0. Define p^* to be equal to q except on extensions s^* of s. There, if σ is the lth string of length $n-1$, then put $\sigma \in p^*(s^*)$ if and only if σ is not accepted by φ_e run relative to u_l. We then have that $p^* \in D_{r,s,k,e}$ as desired. Recall here that $\gamma \cdot r^*$ is incompatible with s^* for all $\gamma \in \mathbb{F}_2$ where $|\gamma| \leq k$, since r^* and s^* extend r and s respectively. □

4 Relativization barriers and largeness notions for sets of polynomial time degrees

Martin's ultrafilter is defined by a game where two players alternate defining the bits of a real, as follows:

Definition 4.1. Let U be the collection of $\equiv_m^{\mathbf{P}}$-invariant Borel sets $A \subseteq \mathcal{P}(2^{<\omega})$ such that player I has a winning strategy in the following game G_A. Players I and II alternate defining which strings are in a language x where on turn n, player I decides membership in x for all strings of length n beginning with a 0, and on turn n, player II decides membership in x for all strings of length n beginning with a 1, so x is the recursive join of the languages played by I and II. Then player I wins the game if $x \in A$.

We now show that U is a σ-complete ultrafilter for $\equiv_m^{\mathbf{P}}$-invariant sets.

Proposition 4.2. *U is a σ-complete ultrafilter on the $\equiv_m^{\mathbf{P}}$-invariant Borel sets.*

Proof. Because A is $\equiv_m^{\mathbf{P}}$-invariant, it is easy to check that for any definition of a recursive join $\oplus$ such that the map $(x, y) \mapsto x \oplus y$ is polynomial time many-one equivalent to our canonical definition in Section 2, the winner of the game does not change. Hence, we can see that $A \in U$ iff $A^c \notin U$ by simply switching the roles of players I and II in G_A.

Similarly, suppose we are given countably many strategies $\sigma_0, \sigma_1, \ldots$ for player I in the games $G_{A_0}, G_{A_1}, \ldots$ respectively. By changing the definition of recursive join, we may assume that in the game G_{A_i}, player I decides membership in x for all string which begin with the string $0^{i+1\frown}1$ of $i+1$ zeroes followed by a one, and player II decides membership in x for all other strings. Now we can use the strategies $\sigma_0, \sigma_1, \ldots$ simultaneously to give a strategy for player I in the game where player I decides x on all strings beginning with 0. This combined strategy shows that player I wins the game with payoff set $\bigcap_i A_i$. Hence, U is σ-complete. □

Next, an easy extension of Theorem 3.1 shows the following:

Theorem 4.3. *Let E be a countable Borel equivalence relation as in Theorem 3.1. Then Definition 4.1 defines an ultrafilter on the Borel E-invariant sets such that for any $A \in U$, $E \restriction A$ is a universal countable Borel equivalence relation.*

Proof. Given a winning strategy σ for player I in the game G_A, and any $y \in \mathcal{P}(2^{<\omega})$, let $\sigma * y$ be the outcome of playing the strategy σ against player II playing y. Then if we replace the definition of $\hat{f}$ in the proof with

$$\hat{f}(x) = \sigma * \left(f(x) \oplus c\left(\hat{f}(\alpha \cdot x) \oplus \hat{f}(\alpha^{-1} \cdot x) \oplus \hat{f}(\beta \cdot x) \oplus \hat{f}(\beta^{-1} \cdot x) \right) \right)$$

then $\hat{f}$ remains a homomorphism of E_∞ into E. This is because for all x, $\hat{f}(\alpha \cdot x)$, $\hat{f}(\alpha^{-1} \cdot x)$, $\hat{f}(\beta \cdot x)$, and $\hat{f}(\beta^{-1} \cdot x)$ are coded in $\hat{f}(x)$ so that they are $O(g)$-time computable from $\hat{f}(x)$. Hence, by induction, if γ is a word of length k, then $\hat{f}(\gamma \cdot x)$ is $O(g^k)$-time computable from $\hat{f}(x)$.

Furthermore, a trivial modification of the proof of Theorem 3.1 shows that with this new definition of $\hat{f}$ and the associated $\hat{p}$, the sets $D_{r,s,k,e}$ are still dense and hence if f is a sufficiently generic continuous function, then $\hat{f}$ will be a Borel reduction. Finally, $\hat{f}$ must be an embedding whose range is contained in the set A, since σ is a winning strategy for player I in G_A. □

Now an easy argument essentially repeating Gill, Baker, and Solovay's proof shows that if $A = \{x : \mathbf{P}^x = \mathbf{NP}^x\}$, then there is a winning strategy for player I in the game G_A; player I simply ensures that the outcome of the game is a language that is polynomial time equivalent to its **PSPACE** completion. Thus, we have the following corollary

Corollary 4.4. *Let $A = \{x : \mathbf{P}^x = \mathbf{NP}^x\}$. Then $\equiv_m^{\mathbf{P}} \restriction A$ and $\equiv_T^{\mathbf{P}} \restriction A$ are universal countable Borel equivalence relations.*

We finish by noting that the ultrafilter U in Definition 4.1 can be combined with Theorem 4.3 to provide an alternate way of proving many of the results in section 3 of [12]; one simply uses them to replace Martin's measure and arithmetic equivalence in that paper.

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