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Descriptive Graph Combinatorics

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1. Introduction

In this article we survey the emerging field of descriptive graph combinatorics. This area has developed in the last two decades or so at the interface of descriptive set theory and graph theory, and it has interesting connections with other areas such as ergodic theory and probability theory.

Our object of study is the theory of definable graphs, usually Borel or analytic graphs on Polish spaces. We investigate how combinatorial concepts, such as colorings and matchings, behave under definability constraints, i.e., when they are required to be definable or well-behaved in the topological or measure theoretic sense.

To illustrate the new phenomena that can arise in the definable context, consider for example colorings of graphs. As usual a *Y-coloring* of a graph $\boldsymbol{G} = (X, G)$, where X is the set of vertices and $G \subseteq X^2$ the edge relation, is a map $c\colon X \to Y$ such that $xGy \implies c(x) \neq c(y)$. An elementary result in graph theory asserts that any acyclic graph admits a 2-coloring (i.e., a coloring as above with $|Y| = 2$). On the other hand, consider a Borel graph $\boldsymbol{G} = (X, G)$, where X is a Polish space and G is Borel (in X^2). A *Borel coloring* of $\boldsymbol{G}$ is a coloring $c\colon X \to Y$ as above with Y a Polish space and c a Borel map. In contrast to the above basic fact, there are acyclic Borel graphs $\boldsymbol{G}$ which admit no Borel countable coloring (i.e., with $|Y| \leq \aleph_0$); see Example 4.14. Moreover for each $n \geq 2$, one can find acyclic Borel graphs $\boldsymbol{G}$, which admit a Borel n-coloring but no Borel m-coloring for any $m < n$; see the first paragraph of Subsection 5,**(B)**. However, sometimes results of classical graph theory have definable counterparts. For example, another standard result in graph theory asserts that every graph of degree at most d admits a $(d + 1)$-coloring. It turns out that every Borel graph of degree at most d actually admits a Borel $(d + 1)$-coloring; see Proposition 5.4. Another interesting example of the interplay between combinatorics and definability is the following. An *edge coloring* of a graph $\boldsymbol{G}$ is a map that assigns to each edge (viewed as a two element set) a color (in some set Y) such that adjacent edges have distinct colors. A classical theorem of Vizing states that in any graph of maximum degree d there is an edge $(d+1)$-coloring. This is in general optimal, although, by a result of König, this upper bound can be lowered to d if the graph is bipartite; see Theorem 6.10, and Remark 6.11. However, it turns out that for Borel graphs of maximum degree d there is always an edge $(2d-1)$-coloring and this is optimal, even among d-regular acyclic graphs; see Theorem 6.12.

Although there were a few isolated results that can now be viewed as belonging to the theory developed here, most notably concerning the measurable chromatic numbers of the unit distance graph in $\mathbb{R}^2$ and related graphs (see Example 5.1), the first systematic study of definable graph combinatorics appears to be the paper Kechris-Solecki-Todorcevic [KST]. Since that time the theory has developed in several directions that will be surveyed in detail in this article. An appealing aspect of this theory is the variety of methods employed in its study, which include methods of effective and classical descriptive set theory, graph theory (including random graphs and the more recently developed graph limits), probability theory, ergodic theory, topological dynamics, ultraproducts, and infinite games (via Borel determinacy).

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2. Outline

We next give an overview of the contents of this survey. It is divided into two parts. Part I, containing Chapters 3-9, deals with graph colorings. Part II, containing Chapters 10-16, deals with matchings. Here is a more detailed description of the contents of each chapter.

Chapter 3 gives the standard preliminaries concerning colorings and graph homomorphisms, both in the classical and the definable context.

Chapter 4 focuses on the dividing line between graphs that have countable vs. uncountable Borel chromatic number. Section 4.1 contains a characterization of when an analytic graph has countable edge Borel chromatic number, while 4.2 contains some basic facts concerning graphs that have countable Borel chromatic number and 4.3 includes examples of interesting Borel graphs that have uncountable Borel chromatic number. In 4.4 we discuss the $\boldsymbol{G}_0$-dichotomy of [KST], which gives the exact obstruction for an analytic graph to have countable Borel chromatic number. It is used in 4.5 to study the structure of the "thin" analytic sets of countable Borel chromatic number (with respect to a given analytic graph of uncountable Borel chromatic number). In 4.6 applications are given to the definable version of Hedetniemi's Conjecture concerning the chromatic number of products of graphs. In 4.7 and 4.8 extensions and additional dichotomy theorems for various classes of analytic graphs, related to our previous dividing line, are given, primarily due to Feng, Lecomte, Louveau and B. Miller. Section 4.9 deals with injective versions of the $\boldsymbol{G}_0$-dichotomy, from [KST] and more recent work of Lecomte and B. Miller. The $\boldsymbol{G}_0$-dichotomy provides a 1-element basis for the quasi-order of analytic graphs with uncountable Borel chromatic number under Borel homomorphism. In 4.10 we discuss results of Lecomte and B. Miller concerning basis problems for such quasi-orders defined by reducibilities as opposed to homomorphisms. Subsection 4.11 presents recent work of Carroy, Lecomte, B. Miller, Soukup and

Zelený, concerning the Borel complexity of countable Borel colorings. B. Miller has in the last few years developed an important theory which provides a unified approach to many dichotomy theorems in descriptive set theory (not necessarily dealing with graphs) based on graph dichotomies extending the $\boldsymbol{G}_0$-dichotomy. This is discussed in 4.12.

Chapter 5 explores a second dividing line, that of graphs with finite vs. countably infinite Borel chromatic number. Section 5.1 discusses interesting examples of Borel graphs with countably infinite Borel chromatic number. In 5.2 we present some basic facts from [KST] concerning upper bounds for the Borel chromatic number of analytic graphs of bounded degree as well as recent results of Conley and B. Miller on Baire measurable chromatic numbers of locally finite analytic graphs. In 5.3 we consider the problem of calculating the Borel chromatic number of the graph generated by finitely many Borel functions and present the known results from [KST], and the more recent work of Conley, Meehan, B. Miller and Palamourdas. In 5.4 and 5.5 we deal with universality results concerning classes of locally countable Borel graphs and Borel functions, including in particular a study of the shift graph on increasing sequences of integers. Finally in 5.6 we discuss the basis problem for the class of Borel graphs with infinite Borel chromatic number and the work of Pequignot, Todorcevic and Vidnyánszky.

Chapter 6 is devoted to the study of graphs with finite Borel chromatic number. In 6.1 we consider an analog of the $\boldsymbol{G}_0$-dichotomy for the property of having Borel chromatic number ≤ 2, due to Carroy, B. Miller, Schrittesser and Vidnyánszky. As we mentioned earlier, a Borel graph of degree $\leq d$ admits a Borel $(d+1)$-coloring. In 6.2 we first discuss the result of Marks, whose proof brought in this area game-theoretic arguments and Borel determinacy, establishing the optimality of this bound for Borel d-regular, acyclic Borel graphs. Next we discuss results of Conley, Kechris, Marks and Tucker-Drob on the extent to which the classical Brooks' bound (that a finite graph of degree $\leq d$ admits a d-coloring except for two obvious exceptions) holds in the Borel, measurable (with respect to a given measure) or Baire measurable context. We conclude with results of Bernshteyn that apply a measurable version of the Lovász local lemma to compute bounds for approximate measurable chromatic numbers. In 6.3 we are concerned with Vizing's Theorem that a finite graph of degree $\leq d$ admits an edge $(d+1)$-coloring. In [KST] it was shown that an analytic graph of degree $\leq d$ admits an edge Borel $(2d-1)$-coloring. We discuss here Marks' result, again proved using game theoretic methods, that this bound is optimal and also results and problems concerning the measure or Baire category framework, including very recent work of Bernshteyn, Csóka, Lippner and Pikhurko. In 6.4 we present results of Conley, Kechris and B. Miller concerning hyperfinite graphs, i.e., locally countable Borel graphs whose connected components define a hyperfinite Borel equivalence relation. A free Borel action of a marked group, i.e., a finitely generated group with a given symmetric finite set of generators, defines a natural Borel graph of bounded degree on the space of the action and 6.5 deals with coloring problems for such graphs in the Borel, measurable or Baire measurable context. Results of Bernshteyn, Conley, Csóka, Gao, Jackson, Kechris, Lippner, Marks, Pikhurko, Seward and Tucker-Drob are presented here. In 6.6 we look at such graphs in the context of ergodic theory, i.e., for measure preserving actions of marked groups on standard measure spaces. We consider combinatorial parameters, such as measurable independence or chromatic numbers, associated with each action and in particular with the Bernoulli actions of free groups. We present here results of Bernshteyn, Conley, Kechris, Lyons, Marks, Nazarov and Tucker-Drob that involve a variety of methods from ergodic theory, spectral theory, random graphs and ultraproducts. In 6.7 we consider connections of the results in 6.6 to problems in probability theory concerning invariant, random colorings in Cayley graphs of marked groups.

In Chapter 7 we survey the possible chromatic numbers in the Borel, measurable and Baire measurable contexts, for various classes of definable graphs, and in Chapter 8 we discuss results concerning other coloring concepts.

Chapter 9 provides a quick introduction to connections with the theory of graph limits of bounded degree finite graphs and contains pointers to the relevant current bibliography.

Chapter 10 contains the basic preliminaries about matchings in graphs. Chapter 11 deals with Marks' result that the classical König Theorem (every regular, bipartite graph admits a perfect matching) fails in the Borel context. On the other hand, in Chapter 12 we discuss theorems of Conley, B. Miller and Marks, Unger on the existence of perfect matchings in the Baire category context and in Chapter 13 we present results of Elek, Lippner, Lyons and Nazarov about matchings in the measure theoretic context. In Chapter 14, in analogy with 6.5, we discuss results of Conley, Csóka, Kechris, Lippner, Lyons, Nazarov, Tucker-Drob on matchings in graphs generated by free, measure preserving actions of marked groups and in Chapter 15 their applications to invariant, random perfect matchings on Cayley graphs. Finally Chapter 16 surveys results connecting equidecomposability and paradoxical decompositions for group actions with matchings. It includes results of Grabowski, Máthé, Pikhurko on equidecomposability of sets in $\mathbb{R}^n$ using Lebesgue measurable pieces and Marks, Unger using Borel pieces, extending the classical Banach-Tarski Paradox and the result of Laczkovich related to his solution of the Tarski Circle Squaring Problem. Also includes work of Marks, Unger on paradoxical decompositions using sets with the property of Baire, which extends the work of Dougherty and Foreman, in connection with their solution to the Marczewski Problem on the existence of Banach-Tarski paradoxical decompositions with pieces that have the property of Baire. It concludes with another proof of the Dougherty and Foreman result.

Part I

COLORINGS

3. Preliminaries on graphs and colorings

Let $L = (R_i)_{i \in I}$ be a relational language, where R_i has arity $n_i \geq 1$. Given two L-structures $\boldsymbol{A} = (X, (R_i^{\boldsymbol{A}})_{i \in I}), \boldsymbol{B} = (Y, (R_i^{\boldsymbol{B}})_{i \in I})$ a *homomorphism* of $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f \colon X \to Y$ such that for each $i \in I$ and $a_1, \ldots, a_{n_i} \in X$,

$$R_i^{\boldsymbol{A}}(a_1, \ldots, a_{n_i}) \implies R_i^{\boldsymbol{B}}(f(a_1), \ldots, f(a_{n_i})).$$

If such a homomorphism exists we write

$$\boldsymbol{A} \preceq \boldsymbol{B}$$

and if an injective homomorphism exists we write

$$\boldsymbol{A} \preceq^{\text{inj}} \boldsymbol{B}.$$

A *reduction* of $\boldsymbol{A}$ to $\boldsymbol{B}$ is a map $f \colon X \to Y$ such that for each $i \in I$ and $a_1, \ldots, a_{n_i} \in X$,

$$R_i^{\boldsymbol{A}}(a_1, \ldots, a_{n_i}) \iff R_i^{\boldsymbol{B}}(f(a_1), \ldots, f(a_{n_i})).$$

If such a reduction exists we write

$$\boldsymbol{A} \leq \boldsymbol{B}$$

and if an injective reduction exists we write

$$\boldsymbol{A} \sqsubseteq \boldsymbol{B}.$$

The notation $\boldsymbol{A} \subseteq \boldsymbol{B}$ means that $\boldsymbol{A}$ is a substructure of $\boldsymbol{B}$. Letting $\boldsymbol{A} \cong \boldsymbol{B}$ denote the isomorphism relation, it follows that an injective reduction of

$\boldsymbol{A}$ into $\boldsymbol{B}$ is an isomorphism of $\boldsymbol{A}$ with a substructure of $\boldsymbol{B}$. For a structure $\boldsymbol{A}$ as above and $Y \subseteq X$ we denote by $\boldsymbol{A}|Y$ the substructure of $\boldsymbol{A}$ with universe Y.

Let Λ be a class of sets in Polish (or standard Borel) spaces, e.g., closed, Borel, analytic, etc. If $\boldsymbol{A} = (X, (R_i^{\boldsymbol{A}})_{i\in I})$ is a structure with X Polish (or standard Borel), we say that $\boldsymbol{A}$ is in Λ if $R_i^{\boldsymbol{A}} \subseteq X^{n_i}$ is in Λ in the product space X^{n_i}, $i \in I$.

Let now Φ be a class of functions between Polish (or standard Borel) spaces, e.g., continuous, Borel, etc. If $\boldsymbol{A} = (X, (R_i^{\boldsymbol{A}})_{i\in I}), \boldsymbol{B} = (Y, (R_i^{\boldsymbol{B}})_{i\in I})$, where X, Y are Polish (or standard Borel) spaces, we write $\boldsymbol{A} \preceq_\Phi \boldsymbol{B}$ if there is a Φ-homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. Similarly for $\preceq_\Phi^{\text{inj}}, \leq_\Phi, \sqsubseteq_\Phi, \cong_\Phi$. We use the subscript c when Φ is the class of continuous functions, B when Φ is the class of Borel functions, BM for the case of Baire measurable functions, μ when Φ is the class of μ-measurable functions (for a Borel measure μ) and UM in the case of universally measurable functions.

We are primarily interested in graphs in this paper. By a *directed graph* we mean a structure $\boldsymbol{D} = (X, D)$, where $D \subseteq (X)^2$, with $(X)^2 = \{(x, y) \in X^2 \colon x \neq y\}$, i.e., D is a non-reflexive binary relation. Let also $D^{-1} = \{(x, y) \colon (y, x) \in D\}$. We usually write xDy instead of $(x, y) \in D$ and let $D(x) = \{y \colon xDy\}$. A *graph* is a structure $\boldsymbol{G} = (X, G)$, where G is a non-reflexive, symmetric binary relation. Thus the class of graphs is contained in the class of directed graphs. Every directed graph $\boldsymbol{D}$ gives rise to the (symmetrized) graph $\boldsymbol{G}_{\boldsymbol{D}} = (X, G_D)$, where

$$G_D = \{(x, y) \colon xDy \text{ or } yDx\} = D \cup D^{-1}.$$

For a graph $\boldsymbol{G} = (X, G)$ we also view G as the set of unordered pairs $\{x, y\}$, where xGy, i.e., the *edges* of $\boldsymbol{G}$.

If $\boldsymbol{D} = (X, D), \boldsymbol{E} = (Y, E)$ are directed graphs, we say that $\boldsymbol{D}$ is a *subgraph* of $\boldsymbol{E}$ if $X \subseteq Y$ and $D \subseteq E$. It is an *induced subgraph* if moreover $D = E \cap X^2$. In the general model theoretic notation, this means that $\boldsymbol{D} \subseteq \boldsymbol{E}$. For a directed graph $\boldsymbol{E} = (Y, E)$ and $X \subseteq Y$, we denote by $\boldsymbol{E}|X$ the induced subgraph with vertex set X.

A *path of length* $n \geq 1$ in a graph $\boldsymbol{G} = (X, G)$ is a sequence of vertices $x_0, x_1, \dots, x_n \in X$ with $x_i G x_{i+1}, \forall i < n$. It is *simple* if $x_0, x_1, \dots, x_n$ are distinct. A *cycle of length* $n \geq 3$ or *n-cycle* in $\boldsymbol{G}$ is a path $x_0, x_1, \dots, x_n$, where $x_n = x_0$. It is *simple* if $x_0, \dots, x_{n-1}$ are distinct. A graph $\boldsymbol{G}$ is *acyclic* if it has no simple n-cycle, for $n \geq 3$.

The *connected components* of a graph $\boldsymbol{G} = (X, G)$ are the equivalence classes of the equivalence relation on X:

$$xE_{\boldsymbol{G}}y \iff \text{there is a } \boldsymbol{G}\text{-path from } x \text{ to } y.$$

For a directed graph $\boldsymbol{D}$, we let $E_{\boldsymbol{D}} = E_{\boldsymbol{G_D}}$.

A graph $\boldsymbol{G} = (X, G)$ is *connected* if it has a single connected component. For such a graph we define the distance $d_{\boldsymbol{G}}(x, y)$, for $x \neq y \in X$, by $d_{\boldsymbol{G}}(x, y) =$ the least n such that there is a $\boldsymbol{G}$-path of length n from x to y (we also put $d_{\boldsymbol{G}}(x, x) = 0$). We extend this notation to arbitrary graphs by letting $d_{\boldsymbol{G}}(x, y) = \infty$ if $\neg xE_{\boldsymbol{G}}y$.

We apply these notions to directed graphs $\boldsymbol{D} = (X, D)$ by postulating that they hold for $\boldsymbol{G_D}$. We can also define the concept of a *directed path* of length $n \geq 1$ in $\boldsymbol{D}$ to be a sequence $x_0, x_1, \ldots, x_n$ with $x_iDx_{i+1}, \forall i < n$, and similarly for directed cycles.

The *out-degree*, $od_{\boldsymbol{D}}(x)$, of a vertex x in $\boldsymbol{D}$ is the cardinality $|D(x)|$ and the *in-degree* of x, $id_{\boldsymbol{D}}(x)$, is the cardinality $|D^{-1}(x)|$, where clearly $D^{-1}(x) = \{y \colon yDx\}$. When $\boldsymbol{G}$ is a graph, we simply talk about the *degree* of x,

$$\deg_{\boldsymbol{G}}(x) = |G(x)|.$$

We also let $\Delta(\boldsymbol{G})$ be the least upper bound of $\deg_{\boldsymbol{G}}(x), x \in X$. The graph is called *bounded degree*, resp., *locally finite, locally countable,* if $\Delta(\boldsymbol{G})$ is finite, resp., each $\deg_{\boldsymbol{G}}(x)$ is finite, $\Delta(\boldsymbol{G}) \leq \aleph_0$. The graph $\boldsymbol{G}$ is *d-regular* if $\deg_{\boldsymbol{G}}(x) = d$ for every vertex x. A directed graph $\boldsymbol{D}$ satisfies one of these properties if the graph $\boldsymbol{G_D}$ does.

An equivalence relation is called *finite*, resp., *countable*, if each equivalence class is finite, resp., countable. Thus $\boldsymbol{G}$ is locally countable iff $E_{\boldsymbol{G}}$ is countable.

Given a family of functions $F_j \colon X \to X, j \in J$, we denote by $\boldsymbol{D}_{(F_j)} = (X, D_{(F_j)})$ the directed graph where

$$xD_{(F_j)}y \iff x \neq y \;\&\; \exists j(F_j(x) = y)$$

and by $\boldsymbol{G}_{(F_j)}$ its symmetrization.

An *independent* set in a directed graph $\boldsymbol{D} = (X, D)$ is a subset $A \subseteq X$ such that $x, y \in A \implies \neg(xDy)$. This coincides with the concept of an independent set in $\boldsymbol{G_D}$. A *Y-coloring* of $\boldsymbol{D}$ is a map $c \colon X \to Y$ such that

$$xDy \implies c(x) \neq c(y),$$

i.e., a homomorphism from $\boldsymbol{D}$ into $\boldsymbol{K}_Y = (Y, (Y)^2)$. Equivalently this means that for each $y \in Y,\ c^{-1}(\{y\})$ is independent. Again this coincides with the notion of coloring of $\boldsymbol{G}_{\boldsymbol{D}}$. The *chromatic number* of $\boldsymbol{D}$, $\chi(\boldsymbol{D})$, is the smallest cardinality of a set Y for which there is a coloring $c\colon X \to Y$. For example, for a graph $\boldsymbol{G}, \chi(\boldsymbol{G}) \leq 2$ iff $\boldsymbol{G}$ has no odd cycles. Such graphs are called *bipartite*. In particular, an acyclic graph is bipartite. Note also that if $\boldsymbol{G}$ is locally countable, then $\chi(\boldsymbol{G}) \leq \aleph_0$.

For every graph $\boldsymbol{G}$, we let $\boldsymbol{L}(\boldsymbol{G})$ be its *line graph*, $\boldsymbol{L}(\boldsymbol{G}) = (G, L(G))$, where G is the set of edges of $\boldsymbol{G}$, viewed as 2-element sets, and

$$\{x,y\}L(G)\{z,w\} \iff \{x,y\} \neq \{z,w\} \ \&\ \{x,y\} \cap \{z,w\} \neq \emptyset,$$

i.e., two edges are connected in $L(G)$ if they have a common vertex. The *edge chromatic number* of $\boldsymbol{G}$, in symbols $\chi'(\boldsymbol{G})$, is the chromatic number of $\boldsymbol{L}(\boldsymbol{G})$.

Given now a directed graph $\boldsymbol{D} = (X, D)$ in a Polish (or standard Borel space X) and a class of functions Φ as before, we define the Φ-*chromatic number* of $\boldsymbol{D}$ as being the smallest cardinality of a Polish space Y for which there is Φ-coloring $c\colon X \to Y$ of $\boldsymbol{D}$. We denote it by $\chi_\Phi(\boldsymbol{D})$. Again this is the same as $\chi_\Phi(\boldsymbol{G}_{\boldsymbol{D}})$. Thus $\chi_\Phi(\boldsymbol{D}) \in \{1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$, because these are the possible cardinalities of Polish spaces. We again use $\chi_B, \chi_{BM}, \chi_\mu$ (for a Borel measure μ) for the case of *Borel, Baire measurable, μ-measurable chromatic numbers*. Clearly $\chi \leq \chi_{BM} \leq \chi_B, \chi \leq \chi_\mu \leq \chi_B$. We also let

$$\chi_M = \sup\{\chi_\mu\colon \mu \text{ is a Borel probability measure}\},$$

which we call the *measure chromatic number*. Similarly we define the number $\chi'_B(\boldsymbol{G})$ for a graph $\boldsymbol{G}$. We also let $\chi'_\mu(\boldsymbol{G})$ be the minimum of $\chi'_B(\boldsymbol{G}|C)$, where C varies over $E_{\boldsymbol{G}}$-invariant conull Borel sets and analogously define $\chi'_{BM}(\boldsymbol{G})$. Finally we put $\chi'_M = \sup_\mu \chi'_\mu$.

When $\boldsymbol{D} = (X, D)$ is a directed graph on a standard Borel space X and μ is a Borel probability measure on X, we also define the *approximate μ-measurable chromatic number of* $\boldsymbol{D}$, in symbols $\chi^{ap}_\mu(\boldsymbol{D})$, to be the smallest cardinality of a standard Borel space Y such that for each $\epsilon > 0$, there is a Borel set $A \subseteq X$ with $\mu(X \setminus A) < \epsilon$ and a μ-measurable coloring $c\colon A \to Y$ of the induced graph $\boldsymbol{D}|A$. Clearly $\chi^{ap}_\mu(\boldsymbol{D}) \leq \chi_\mu(\boldsymbol{D})$. Again $\chi^{ap}_\mu(\boldsymbol{D}) = \chi^{ap}_\mu(\boldsymbol{G}_{\boldsymbol{D}})$. We define analogously the *approximate measure chromatic number* χ^{ap}_M.

Finally, the *μ-independence number* of $\boldsymbol{D}$, in symbols $i_\mu(\boldsymbol{D})$, is the supremum of the μ-measures of $\boldsymbol{D}$-independent Borel sets. Clearly $i_\mu(\boldsymbol{D}) = i_\mu(\boldsymbol{G_D})$. Moreover we have the following obvious inequality:

$$\chi_\mu^{ap}(\boldsymbol{D}) \geq \frac{1}{i_\mu(\boldsymbol{D})},$$

if $i_\mu(\boldsymbol{D}) > 0$, while $\chi_\mu^{ap}(\boldsymbol{D}) = 2^{\aleph_0}$ if $i_\mu(\boldsymbol{D}) = 0$. Thus graphs with small independence number have large (approximate) measurable chromatic number. Moreover, by an easy exhaustion argument, we have that $\chi_\mu^{ap}(\boldsymbol{D}) \leq \aleph_0$ iff there is a Borel set $Y \subseteq X$ with $\mu(Y) = 1$ such that $\chi_B(\boldsymbol{D}|Y) \leq \aleph_0$ iff for every Borel $A \subseteq X$ with $\mu(A) > 0$, there is a $\boldsymbol{D}$-independent Borel $B \subseteq A$ with $\mu(B) > 0$.

Remark 3.1. Geschke [G11] also considers *weak Borel chromatic numbers* which correspond to arbitrary colorings but in which the points having the same color form a Borel set.

4. Countable vs. uncountable Borel chromatic numbers

4.1 Edge chromatic numbers

We start with the following basic fact:

Proposition 4.1 ([KST, 4.10]). *Let $\boldsymbol{G} = (X, G)$ be an analytic graph on a standard Borel space X. Then the following are equivalent*

(i) $\chi'_B(\boldsymbol{G}) \leq \aleph_0$,

(ii) $\boldsymbol{G}$ *is locally countable.*

Remark 4.2. Proposition 4.1 is proved in [KST, 4.10] for Borel graphs but easily extends to analytic graphs, since every analytic graph $\boldsymbol{G}$ (on a standard Borel space) which is locally countable is a subgraph of a locally countable Borel graph $\boldsymbol{H}$ with the same set of vertices. This follows from the First Reflection Theorem (see [K95, 35.10]).

Note that if $\boldsymbol{G}$ is Borel locally countable, $N \in \{1, 2, \dots, \mathbb{N}\}$, where we identify $n \geq 1$ with $n = \{0, \dots, n-1\}$, and $c \colon G \to N$ is surjective and a Borel edge coloring, then for each $n \in N, c^{-1}(\{n\})$ is the graph $\boldsymbol{G}_{T_n}$ of a Borel involution T_n of X and thus $\boldsymbol{G} = \boldsymbol{G}_{(T_n)}$. Thus Proposition 4.1 for Borel $\boldsymbol{G}$ is equivalent to the statement that every locally countable Borel graph is generated by a countable family of Borel involutions.

There is a related proposition for directed graphs. Let $\boldsymbol{D} = (X, D)$ be a directed graph. Its *shift graph* (also called the *directed line graph* of $\boldsymbol{D}$) is the directed graph $\boldsymbol{sD} = (D, sD)$, where sD consists of all pairs $((x, y), (y, z))$ with xDy, yDz.

Proposition 4.3 ([KST, 4.11]). *Let $\boldsymbol{D}$ be an analytic directed graph on a standard Borel space X. Then $\chi_B(\boldsymbol{sD}) \leq \aleph_0$.*

Remark 4.4. Similar comments as in Remark 4.2 apply here.

4.2 Countable Borel chromatic numbers

We first note the following reformulation of the existence of a countable Borel coloring (i.e., $\chi_B \leq \aleph_0$).

Proposition 4.5 ([KST, 4.3, 4.4]). *Let $\boldsymbol{D} = (X, D)$ be an analytic directed graph on a standard Borel space X. Then $\chi_B(\boldsymbol{D}) \leq \aleph_0$ iff there is a Polish topology σ generating the Borel structure of X such that $\forall x(x \notin \overline{D(x)}^{\sigma})$.*

Proof. $\implies$: If $c\colon X \to \mathbb{N}$ is a Borel coloring of $\boldsymbol{D}$, then there is a Polish topology generating the Borel structure of X in which c is continuous (with $\mathbb{N}$ discrete). Then $A_n = c^{-1}(\{n\})$ is σ-clopen and if $x \in A_n$, then $A_n \cap D(x) = \emptyset$, so x is not in the closure $\overline{D(x)}^{\sigma}$.

$\impliedby$: Let $\{U_n\}$ be an open basis for the topology σ. Let

$$P(x, n) \iff x \in U_n \;\&\; U_n \cap D(x) = \emptyset.$$

Then $P \subseteq X \times \mathbb{N}$ is co-analytic and $\forall x \exists n P(x, n)$, so by the Number Uniformization Property for co-analytic sets (see [K95, 35.1]), there is a Borel function $f\colon X \to \mathbb{N}$ such that $\forall x P(x, f(x))$. Then $c(x) = U_{f(x)}$ is a countable Borel coloring of $\boldsymbol{D}$. $\square$

Corollary 4.6 ([KST, 4.5]). *Let $\boldsymbol{D} = (X, D)$ be an analytic directed graph on a standard Borel space X such that $\forall x(D(x)$ is finite), i.e., $\boldsymbol{D}$ has locally finite out-degree. Then $\chi_B(\boldsymbol{D}) \leq \aleph_0$. In particular, this holds for locally finite Borel graphs $\boldsymbol{G}$ and graphs of the form $\boldsymbol{G} = \boldsymbol{G}_{F_1,\dots,F_n}$, $F_i\colon X \to X$ Borel, $1 \leq i \leq n$.*

Recall that if X is a standard Borel space, $R \subseteq X^n, n \geq 1$, and Λ is a class of sets in Polish spaces closed under continuous preimages, we say that R is *potentially* Λ if there is a Polish topology σ generating the Borel structure of X such that R is in Λ in the space X^n with the product σ-topology. Equivalently this means that there is a Λ-relation $S \subseteq Y^n, Y$ a Polish space, with $R \leq_B S$. We now have

Corollary 4.7. *Let $\boldsymbol{D} = (X, D)$ be a directed graph on a standard Borel space which is potentially closed. Then $\chi_B(\boldsymbol{D}) \leq \aleph_0$.*

It is not hard to see that the converse is not true.

Example 4.8. First notice that if X is an uncountable Polish space, then the diagonal $\{(x, y) \in X^2 \colon x = y\}$ is not potentially open, i.e., it is not the union of countably many Borel rectangles. It follows that if X, Y are uncountable standard Borel spaces, then there is Borel subset of $X \times Y$ which is not the union of countably many Borel rectangles. Now let X be again an uncountable standard Borel space and let $X = A_1 \sqcup A_2$ be a decomposition of X into two uncountable Borel sets. Let $B \subseteq A_1 \times A_2$ be such that B is not the union of countably many Borel rectangles. Then $\boldsymbol{G} = (X, G)$, where $G = X^2 \setminus (B \cup B^{-1} \cup (A_1)^2 \cup (A_2)^2)$ is a Borel graph with $\chi_B(\boldsymbol{G}) = 2$ (as A_1, A_2 are $\boldsymbol{G}$-independent) but G is not potentially closed (since $X^2 \setminus G$ is not potentially open).

It was shown however in Lecomte-Miller [LM, 5.7] that the converse to Corollary 4.7 holds if $\boldsymbol{D} = (X, D)$ is a graph in a Polish space X with $\overline{D} \setminus D \subseteq \{(x, y) \colon x = y\}$.

Also Lecomte-Miller [LM, 5.8] show the following: Let X be a standard Borel space and $R \subseteq X^2$ be Borel. Define the following directed graph $\boldsymbol{D}_R = (X^2 \setminus R, D_R)$, where

$$(x, y) D_R (z, w) \iff xRw.$$

Then R is potentially closed iff $\chi_B(\boldsymbol{D}_R) \leq \aleph_0$.

We next mention some very useful consequences of the existence of countable Borel colorings.

First recall that given a graph $\boldsymbol{G} = (X, G)$ a *kernel* of $\boldsymbol{G}$ is a maximal independent set $A \subseteq X$. This means that $x \neq y \in A \implies (x, y) \notin G$ and for any $x \notin A$, there is $y \in A$ with $(x, y) \in G$. Below for any $A \subseteq X$, we let $N_{\boldsymbol{G}}(A) = \{y \colon \exists x \in A(xGy)\}$. Thus $G(x) = N_G(\{x\})$, for $x \in X$.

Proposition 4.9 ([KST, 4.2])**.** *Let $\boldsymbol{G} = (X, G)$ be a graph on a standard Borel space such that for any Borel $A \subseteq X$, the set $N_{\boldsymbol{G}}(A)$ is Borel (e.g., if $\boldsymbol{G}$ is Borel locally countable). If $\chi_B(\boldsymbol{G}) \leq \aleph_0$, then every Borel independent set can be extended to a Borel kernel.*

Let E be an equivalence relation on a set X. Denote by $[E]^{<\infty}$ the set of finite nonempty subsets of X which are contained in a single E-class. If E is a Borel equivalence relation on a standard Borel space X, $[E]^{<\infty}$ forms a standard Borel space. The *intersection graph* $\boldsymbol{G}_I = ([E]^{<\infty}, G_I)$ (relative to E), with vertex set $[E]^{<\infty}$, is defined by

$$AG_IB \iff A \neq B \;\&\; A \cap B \neq \emptyset.$$

Proposition 4.10 ([KM, 7.3],[CM2, Proposition 2])**.** *The intersection graph of a countable Borel equivalence relation has countable Borel chromatic number, $\chi_B(\boldsymbol{G}_I) \leq \aleph_0$.*

Note that Proposition 4.10 generalizes Proposition 4.1, when applied to $E = E_{\boldsymbol{G}}$.

One further application of Propositions 4.9 and 4.10 is the existence of appropriately maximal finite (partial) subequivalence relations of a given countable Borel equivalence relation.

Let E be an equivalence relation on a space X. An *fsr* (finite partial subequivalence relation) is an equivalence relation F on a set $Y \subseteq X$ such that $F \subseteq E$ and each F-class is finite. Given $\Psi \subseteq [E]^{<\infty}$ we say that an fsr F on $Y \subseteq X$ is Ψ*-maximal* if $[y]_F \in \Psi, \forall y \in Y$, and if $A \in [E]^{<\infty}$ is a finite set disjoint from Y, then $A \notin \Psi$. We now have:

Proposition 4.11 ([KM, 7.3])**.** *If E is a countable Borel equivalence relation and $\Psi \subseteq [E]^{<\infty}$ is Borel, then there is a Borel Ψ-maximal fsr.*

Remark 4.12. We record here some reformulations (in certain cases) of the condition that the Baire measurable chromatic number is countable. Let $\boldsymbol{D} = (X, D)$ be a directed graph on a Polish space X with $\chi(\boldsymbol{D}) \leq \aleph_0$. Then the following are equivalent:

(i) $\chi_{BM}(\boldsymbol{D}) \leq \aleph_0$,

(ii) There is a dense G_δ set $Y \subseteq X$ and a countable partition of Y into relatively open independent sets.

(iii) There is a dense G_δ set $Y \subseteq X$ such that for any nonempty relatively open $U \subseteq Y$, there is a nonempty relatively open $V \subseteq U$ which is independent.

((i) $\implies$ (ii) follows from the fact that any set with the Baire property becomes open relative to a dense G_δ set, and (ii) $\implies$ (iii) is obvious. For (iii) $\implies$ (i), let $\{U_n\}$ be a maximal collection of pairwise disjoint, relatively open, independent subsets of Y. Then $\bigcup_n U_n$ is dense open in Y, so comeager in X. Using $\chi(\boldsymbol{D}) \leq \aleph_0$, let $\{V_n\}$ be a countable decomposition of $X \setminus \bigcup_n U_n$ into independent sets. Then $\{U_n, V_n\}$ is a countable decomposition of X into independent sets with the Baire property.)

In case $\boldsymbol{D} = \boldsymbol{D}_{(F_n)}$, where $F_n \colon X \to X$ are homeomorphisms of a Polish space X (so that $\chi(\boldsymbol{D}) \leq \aleph_0$), then we can add another equivalence, analogous to Proposition 4.5.

(iv) The set $\{x \colon x \notin \overline{D(x)}\}$ is comeager.

(By Proposition 4.5 and the fact that $\chi(\boldsymbol{D}) \leq \aleph_0$, we have that (iv) $\implies$ (i). To see that (i) $\implies$ (iv), assume that the Borel set $A = \{x \colon x \in \overline{D(x)}\}$ is not meager. Then find an open nonempty set $U \subseteq X$, and, using $\chi_{BM}(\boldsymbol{D}) \leq \aleph_0$, an independent set $B \subseteq X$, with the Baire property, such that $C = A \cap B$ is comeager in U. If $x \in C \cap U$, then $D(x) \cap U \neq \emptyset$, so there is F_n with $F_n(U) \cap U \neq \emptyset$. But $F_n(C)$ is comeager in $F_n(U)$, so $C \cap F_n(C) \neq 0$, contradicting the independence of C.)

4.3 Examples of graphs with uncountable Borel chromatic number

We discuss here some examples of Borel, locally countable, bipartite or acyclic graphs that have uncountable Borel chromatic number. Such graphs have of course chromatic number equal to 2. These examples are based on the simple observation that if the μ-measurable chromatic number of a graph is countable, then there must exist a μ-positive measure independent set, and similarly for category. The graphs in these examples are such that all independent sets are null or meager.

Example 4.13 ([Sz], [T85])**.** These are the first examples in the literature that we are aware of. Let $X = \mathbb{R}$ and let $\boldsymbol{G}_i = (X, G_i)$, $i = 1, 2$, with $xG_1y \iff \exists n \in \mathbb{N}(|x - y| = a_n)$, where $a_n \to 0$ are positive, linearly independent over the rationals (Székely) and $xG_2y \iff \exists k \in \mathbb{Z}(|x - y| = 3^k)$ (Thomas). These are Borel, locally countable bipartite graphs (they have no odd cycles – but have 4-cycles), so that $\chi(\boldsymbol{G}_i) = 2$ but $\chi_{BM}(\boldsymbol{G}_i) = \chi_\mu(\boldsymbol{G}_i) = \chi_B(\boldsymbol{G}_i) = 2^{\aleph_0}$, where $\mu =$ Lebesgue measure. This is because if $A \subseteq \mathbb{R}$ is non-meager with the Baire property or has positive measure, then $A - A = \{x - y \colon x, y \in A\}$ contains an open interval around 0, so there are $x, y \in A$ with $x G_i y$, i.e., A is not independent. Other examples of a similar nature are discussed in [So, §46].

Example 4.14 ([KST, 3.1])**.** Let S_∞ be the Polish group of all permutations on $\mathbb{N}$ and let $(g_n) \in (S_\infty)^{\mathbb{N}}$ be such that $\{g_n \colon n \in \mathbb{N}\}$ is dense in S_∞ and generates a free subgroup of S_∞. Let $F_n \colon S_\infty \to S_\infty$ be given by $F_n(g) = g_n g$. Then $\boldsymbol{G}_{(F_n)}$ is Borel, locally countable and acyclic, so $\chi(\boldsymbol{G}_{(F_n)}) = 2$, but (by an argument similar to that of the previous example) $\chi_{BM}(\boldsymbol{G}_{(F_n)}) = \chi_B(\boldsymbol{G}_{(F_n)}) = 2^{\aleph_0}$.

A variant of this example is given in [CK, 2.2, a)] to produce Borel, acyclic, locally countable graphs $\boldsymbol{G}$ with $\chi_\mu(\boldsymbol{G}) = \chi_{BM}(\boldsymbol{G}) = \chi_B(\boldsymbol{G}) = 2^{\aleph_0}$ (for an appropriate Borel probability measure μ).

Example 4.15 ([CK, 2.2, b)])**.** Let $\Gamma = \mathbb{F}_\infty =$ the free group with $\aleph_0$ many free generators $\{\gamma_n \colon n \in \mathbb{N}\}$. Let $S = \{\gamma_n^{\pm 1} \colon n \in \mathbb{N}\}$. Let $a \colon \Gamma \times X \to X$ be a Borel free, measure preserving, mixing action of Γ on (X, μ), a standard (probability) measure space. We write $\gamma \cdot x = a(\gamma, x)$ if a is understood. For example, a could be the shift action of Γ on 2^Γ, restricted to its free part (i.e., the set of $x \in 2^\Gamma$ for which $\gamma \cdot x \neq x, \forall \gamma \in \Gamma \setminus \{1\}$), with μ the usual product measure on 2^Γ. Let $\boldsymbol{G}(S, a)$ be the "Cayley graph" associated with this action: $\boldsymbol{G}(S, a) = (X, G(S, a))$, where $x G(S, a) y \iff \exists \gamma \in S(\gamma \cdot x = y)$. This is a Borel, locally countable, acyclic graph, so $\chi(\boldsymbol{G}(S, a)) = 2$, but $\chi_\mu(\boldsymbol{G}(S, a)) = \chi_B(\boldsymbol{G}(S, a)) = 2^{\aleph_0}$. This is because, by the mixing property of the shift action, if $A \subseteq X$ has positive measure, then for some n, $\gamma_n \cdot A \cap A$ has positive measure, so there are $x, y \in A$ with $x G(S, a) y$.

It was shown in [CK, 4.6], that this fails if we only assume that the action is ergodic, in fact there are free, measure preserving, ergodic actions of $\mathbb{F}_\infty$ with $\chi_\mu(\boldsymbol{G}(S, a)) = 2$. However it is easy to see that if a is a free, measure preserving, weakly mixing action of $\mathbb{F}_\infty$, then $\chi_\mu(\mathbf{G}(S, a)) > 2$. Otherwise there would be a Borel set $A \subseteq X$ with $0 < \mu(A) < 1$ such that $\gamma_n \gamma_m \cdot A = A$ for $m, n \in \mathbb{N}$, and therefore A would be Γ-invariant, where $\Gamma \leq \mathbb{F}_\infty$ is the index 2 subgroup of $\mathbb{F}_\infty$ which is the kernel of the homomorphism $\pi \colon \mathbb{F}_\infty \to \mathbb{Z}/2\mathbb{Z}$ that sends the generators to 1. Since, by weak mixing, Γ acts ergodically, we have a contradiction. There are examples of free, weakly mixing actions a of $\mathbb{F}_\infty$ with $\chi_\mu(\boldsymbol{G}(S, a)) = 3$ (see [LN, Section 5]).

Example 4.16 ([KST, 6.1])**.** Let $(s_n)_{n\in\mathbb{N}}$ be a sequence of elements of $2^{<\mathbb{N}}$, the set of finite binary sequences, such that (i) $|s_n| = n$ and (ii) (s_n) is dense, i.e., $\forall s \in 2^{<\mathbb{N}} \exists n(s \subseteq s_n)$. Define the graph $\boldsymbol{G}_0^{(s_n)} = \boldsymbol{G}_0 = (2^{\mathbb{N}}, G_0)$ as follows

$$xG_0y \iff \exists n(x|n = y|n = s_n \;\&\; x(n) \neq y(n) \;\&\; \forall m > n(x(m) = y(m)).$$

Then $\boldsymbol{G}_0$ is acyclic (so $\chi(\boldsymbol{G}_0) = 2$) but $\chi_{BM}(\boldsymbol{G}_0) = \chi_B(\boldsymbol{G}_0) = 2^{\aleph_0}$. Indeed if $A \subseteq 2^{\mathbb{N}}$ is a non-meager set with the Baire property, there is some basic nbhd $N_t = \{x \in 2^{\mathbb{N}} : t \subseteq x\}$, $t \in 2^{<\mathbb{N}}$, such that A is comeager on N_t. By density we can assume that $t = s_n$ for some n. Since $\pi(t\hat{}i\hat{}x) = t\hat{}(1-i)\hat{}x$ is a homeomorphism of N_t, $\pi(A\cap N_t)\cap(A\cap N_t) \neq \emptyset$, so A is not independent.

Remark 4.17. Notice that in the preceding proof that $\chi_{BM}(\boldsymbol{G}_0) = 2^{\aleph_0}$ one only needs that (s_n) is dense below some $t \in 2^{<\mathbb{N}}$, i.e., $\exists t \forall s \supseteq t \exists n(s \subseteq s_n)$. Conversely, using Remark 4.12, it is easy to see that if $\chi_{BM}(\boldsymbol{G}_0^{(s_n)}) = 2^{\aleph_0}$, then (s_n) must be dense below some $t \in 2^{<\mathbb{N}}$. If we choose $s_n = 0^n$, then $\chi_B(\boldsymbol{G}_0^{(s_n)}) = \aleph_0$ (see [KST, §6 (B)]), while if we take $s_n = 0^{n-1}\hat{}1$ (for $n > 0$), then $\chi_B(\boldsymbol{G}_0^{(s_n)}) = 2$.

In [KST, §6 (C)], it was mentioned that $\chi_{UM}(\boldsymbol{G}_0) = 2^{\aleph_0}$ (where χ_{UM} is the *universally measurable chromatic number*). This is not correct and in fact Miller [M08] proved that $\chi_M(\boldsymbol{G}_0) = 3$ (and $\chi_\mu(\boldsymbol{G}_0) = 3$, for the usual product measure μ on $2^{\mathbb{N}}$), and assuming CH (or even add(null)= $2^{\aleph_0}$) ,we have $\chi_{UM}(\boldsymbol{G}_0) = 3$. In fact, Miller [M08, 3.1] showed, more generally, that if $\boldsymbol{G} = (X, G)$ is an acyclic, locally countable Borel graph and $E_{\boldsymbol{G}}$ is hyperfinite, then $\chi_M(\boldsymbol{G}) \leq 3$.

On the other hand, if one considers the bigger graph $\boldsymbol{G}_0' = (2^{\mathbb{N}}, G_0')$, where $G_0' \supseteq G_0$ is defined by $xG_0'y \iff \exists! n(x(n) \neq y(n))$ (see [CK, 3.7]), then $\boldsymbol{G}_0'$ has no odd cycles, so $\chi(\boldsymbol{G}_0') = 2$, but $\chi_B(\boldsymbol{G}_0') = \chi_{BM}(\boldsymbol{G}_0') = \chi_\mu(\boldsymbol{G}_0') = 2^{\aleph_0}$ (for the usual product measure μ on $2^{\mathbb{N}}$) by a simple density argument.

Note that the equivalence relation $E_{\boldsymbol{G}_0}$ induced by $\boldsymbol{G}_0$ is E_0, where we put $xE_0y \iff \exists n \forall m \geq n(x(m) = y(m))$, and the same is true for $\boldsymbol{G}_0'$.

The graph $\boldsymbol{G}_0$ plays a special role, since as we will see next it is the "minimum" graph with uncountable Borel chromatic number.

4.4 The G_0-dichotomy

Recall the notion of homomorphism for graphs discussed in Section 2. Note that $G \preceq H \implies \chi(G) \leq \chi(H)$ and similarly $G \preceq_B H \implies \chi_B(G) \leq \chi_B(H)$. We now have:

Theorem 4.18 (The $\boldsymbol{G}_0$-dichotomy, [KST, 6.3]). *For any analytic graph $\boldsymbol{G} = (X, G)$ on a Polish space X, exactly one of the following holds:*

(i) $\chi_B(\boldsymbol{G}) \leq \aleph_0$,

(ii) $\boldsymbol{G}_0 \preceq_c \boldsymbol{G}$.

Note that alternative (i) can be also stated as: $\boldsymbol{G} \preceq_B \boldsymbol{K}_{\mathbb{N}} = (\mathbb{N}, (\mathbb{N})^2)$. Thus the $\boldsymbol{G}_0$-dichotomy can be stated as follows: For any analytic graph $\boldsymbol{G}$, we have that $\boldsymbol{G} \not\preceq_B \boldsymbol{K}_{\mathbb{N}} \iff \boldsymbol{G}_0 \preceq_B \boldsymbol{G}$, i.e, $(\boldsymbol{K}_{\mathbb{N}}, \boldsymbol{G}_0)$ is a *dual pair* in the terminology of [HN, Section 3.8]. The following is also trivially a dual pair: $(\boldsymbol{K}_1, \boldsymbol{K}_2)$, where $\boldsymbol{K}_n$ is the *n-clique*.

Problem 4.19. *Are there any other dual pairs for $\preceq_B$ among analytic (or Borel) graphs?*

It follows from Theorem 4.18 that having countable Borel chromatic number is quite robust. A subset $A \subseteq X$ of a standard Borel space X is called *globally Baire* if for any Borel function $f\colon Y \to X, Y$ a Polish space, $f^{-1}(A)$ has the property of Baire. The globally Baire sets form a σ-algebra. We call a function $f\colon X \to Y$ on standard Borel spaces *globally Baire measurable* if the preimage of any Borel set is globally Baire. We denote by χ_{GB} the corresponding chromatic number for the class of globally Baire measurable colorings. Then we have for any analytic graph $\boldsymbol{G}$ on a standard Borel space $X : \chi_B(\boldsymbol{G}) \leq \aleph_0 \iff \chi_{GB}(\boldsymbol{G}) \leq \aleph_0$.

Similarly the condition $\boldsymbol{G}_0 \preceq_c \boldsymbol{G}$ in (ii) can be replaced by $\preceq_{BM}$ but, contrary to the statement in [KST, §6 (C)], not by $\preceq_{UM}$. This follows again from Miller [M08, 3.2], which implies, under CH (or even add(null))$= 2^{\aleph_0}$), that if $\boldsymbol{K}_3$ is the 3-clique, then both $\chi_B(\boldsymbol{K}_3) = 3$ and $\boldsymbol{G}_0 \preceq_{UM} \boldsymbol{K}_3$.

Finally, in relation to the $\boldsymbol{G}_0$-dichotomy, we can ask whether there are other interesting examples of Borel graphs which are minima in $\preceq_B$ among analytic graphs of uncountable Borel chromatic number. Note that any such graph must be bipartite and have $\chi_M \leq 3$.

4.5 Analytic sets that have countable Borel chromatic number

Let $\boldsymbol{G} = (X, G)$ be an analytic graph on a Polish space X *with uncountable Borel chromatic number*. For an analytic set $A \subseteq X$, we interpret $\chi_B(\boldsymbol{G}|A) \leq \aleph_0$ as meaning that A can be partitioned in countably many Borel, relative to A, $\boldsymbol{G}$-independent sets. If we view $\boldsymbol{G}|A$ as an analytic graph on X, i.e., as the graph $(X, G \cap A^2)$, this is equivalent to stating that the Borel chromatic number of this graph is countable. Put

$$\mathcal{I}_{\aleph_0} = \mathcal{I}^{\boldsymbol{G}}_{\aleph_0} = \{A \subseteq X \colon A \text{ is analytic } \& \ \chi_B(\boldsymbol{G}|A) \leq \aleph_0\}.$$

Using the $\boldsymbol{G}_0$-dichotomy it is easy to see that $\mathcal{I}_{\aleph_0}$ is a σ-ideal of analytic sets. Next note that, as a subset of the power set of X, $\mathcal{I}_{\aleph_0}$ is coanalytic on analytic, i.e., for any Polish space Y and any analytic set $B \subseteq Y \times X$ the set

$$\hat{B} = \{y \in Y \colon B_y \in \mathcal{I}\}$$

is coanalytic. To see this notice that

$$y \notin \hat{B} \iff \exists f \in C(2^{\mathbb{N}}, X) \forall^* \alpha \in 2^{\mathbb{N}} \forall \beta G_0 \alpha (f(\alpha), f(\beta) \in B_y \ \& \ f(\alpha) G f(\beta)).$$

Here $C(2^{\mathbb{N}}, X)$ is the Polish space of all continuous functions $f \colon 2^{\mathbb{N}} \to X$ with the uniform metric (with respect to some fixed metric on X), see [K95, 4.19]. Also for any topological space Z, $\forall^* z \in Z P(z)$ means that $\{z \in Z \colon P(z)\}$ is comeager in Z. Granting this equivalence, and using the fact that the map $(f, \alpha) \to f(\alpha)$ is continuous and the category quantifier $\forall^*$ preserves analyticity (see [K95, 29.22]), it follows that $\hat{B}$ is coanalytic.

To prove the equivalence, notice that the direction $\implies$ is obvious from the $\boldsymbol{G}_0$-dichotomy. To establish the direction $\impliedby$, assume that $y \in \hat{B}$ but f satisfies the right-hand side. Let $A = \{\alpha \in 2^{\mathbb{N}} \colon \forall \beta G_0 \alpha (f(\alpha), f(\beta) \in B_y \ \& \ f(\alpha) G f(\beta))\}$. This is comeager, so contains a dense G_δ set C which is E_0-invariant and therefore $\boldsymbol{G}_0|C \preceq_B \boldsymbol{G}|B_y$ (via f). Since $\chi_B(\boldsymbol{G}_0|C)$ is uncountable, this contradicts that $y \in \hat{B}$.

From the First Reflection Theorem (see [K95, 35.10], it follows then that if A is an analytic set in $\mathcal{I}_{\aleph_0}$, then there is a Borel set $B \supseteq A$ which is also in $\mathcal{I}_{\aleph_0}$. Moreover

$$I_{\aleph_0} = \{K \subseteq X \colon K \text{ is compact } \ \& \ K \in \mathcal{I}_{\aleph_0}\}$$

is a coanalytic σ-ideal of compact sets (see [KLW] and [MZ] for the theory of such σ-ideals). From the $\boldsymbol{G}_0$-dichotomy it also follows immediately that $\mathcal{I}_{\aleph_0}$ has the *inner approximation property* (see [KLW, 3.2]), i.e., any analytic set not in $\mathcal{I}_{\aleph_0}$ contains a compact set not in $\mathcal{I}_{\aleph_0}$. In particular, $I_{\aleph_0}$ is *calibrated* (see [KLW, 3.2]), i.e., if K_n, K are compact subsets of X, $K_n \in I_{\aleph_0}$ and every compact subset of $K \setminus \bigcup_n K_n$ is in $I_{\aleph_0}$, then $K \in I_{\aleph_0}$. (In fact $I_{\aleph_0}$ is *strongly calibrated*, see [KLW, 3.3]). Next we calculate the descriptive complexity of $I_{\aleph_0}$ as a subset of the Polish space $K(X)$ of compact subsets of X.

Proposition 4.20. *The set $I_{\aleph_0}$ is complete coanalytic.*

Proof. By [KLW, §1, Theorem 7] it is enough to show that $I_{\aleph_0}$ is not G_δ. We will define a continuous function $g\colon 2^{\mathbb{N}} \to K(2^{\mathbb{N}})$ such that identifying $2^{\mathbb{N}}$ with the power set of $\mathbb{N}$, we have:

(i) If $A \subseteq \mathbb{N}$ is finite, then $g(A)$ is finite.

(ii) If $A \subseteq \mathbb{N}$ is infinite, then $\boldsymbol{G}_0|g(A)$ has uncountable Borel chromatic number.

Granting this let $f\colon 2^{\mathbb{N}} \to X$ be a continuous homomorphism from $\boldsymbol{G}_0$ into $\boldsymbol{G}$ and let $h\colon 2^{\mathbb{N}} \to K(X)$ be defined by $h(A) = \{f(x)\colon x \in g(A)\}$. Then h is continuous, and A is finite iff $h(A) \in I_{\aleph_0}$, so $I_{\aleph_0}$ is not G_δ.

To define g, consider the sequence (s_n) for $\boldsymbol{G}_0$. We will assign to each $A \subseteq \mathbb{N}$ of natural numbers a tree T_A of binary sequences and take $g(A) = [T_A]$.

Define first a strictly increasing sequence $(k_i)_{i\in\mathbb{N}}$ of natural numbers as follows: Set $k_0 = 0$, and let $k_{i+1} > k_i$ be the least number such that for every finite binary string t of length $\leq k_i$, there is an element of (s_n) extending t of length $< k_{i+1} - 1$. Now define T_A to be the tree which exactly branches at all levels between k_i and k_{i+1} for every $i \in A$. That is, T_A is the set of strings t such that for all $i \notin A$ and for every $k < |t|$ such that $k_i \leq k < k_{i+1}$ we have $t(k) = 0$. □

Next we show that the σ-ideal $I_{\aleph_0}$ is not *thin* (in the sense of [KLW, 3.3]), i.e., there exist continuum many pairwise disjoint compact sets $(A_x)_{x\in 2^{\mathbb{N}}}$ such that $\chi_B(\boldsymbol{G}|A_x) = 2^{\aleph_0}$ for all $x \in 2^{\mathbb{N}}$.

Proposition 4.21. *The σ-ideal $I_{\aleph_0}$ is not thin.*

Proof. We use the following result of Zelený, see [MZ, Theorem 3.35]: Every calibrated, thin, coanalytic σ-ideal of compact sets is G_δ. Then by Proposition 4.20 and the paragraph preceding it, $I_{\aleph_0}$ is not thin. □

Recall that a *basis* for a σ-ideal I of compact sets is a hereditary subset B of I such that for every $K \in I$ there is a sequence (K_n) of sets in B with $K = \bigcup_n K_n$. We also say that I has the *covering property* if for any analytic set A for which all its compact subsets are in I, there is a sequence (K_n) of elements of I such that $A \subseteq \bigcup_n K_n$. A result of Debs-Saint Raymond [DSR] (see also [KL, page 208]) asserts that for every coanalytic, calibrated σ-ideal I of compact sets, which is *locally non-Borel* (i.e., for compact F not in I, the set $I \cap K(F)$ is not Borel), if I has a Borel basis, then I has the covering property. Note that $I_{\aleph_0}$ satisfies all the hypotheses of this theorem, and thus if it has a Borel basis, it also has the covering property.

Problem 4.22. *Does the ideal $I_{\aleph_0}$ have a Borel basis? Does it have the covering property?*

Using the result of Uzcátegui [U, Theorem 3.2] and Theorem 4.42 below, it can be shown that the answer to Problem 4.22 is negative for any $I^{\boldsymbol{G}}_{\aleph_0}$, where $\boldsymbol{G}$ is a locally countable analytic graph (in particular for $\boldsymbol{G}_0$). On the other hand, William Chan pointed out that there is a positive answer for any analytic graph in which every Borel independent set is countable (e.g., the complete graph on $2^{\mathbb{N}}$), since in this case $I_{\aleph_0}$ is the σ-ideal of countable compact sets.

We also mention that Zapletal [Za, 4.7.20] shows that the forcing associated with the ideal $\mathcal{I}^{\boldsymbol{G}_0}_{\aleph_0}$ is not proper. Finally Zapletal [Za1, Section 4] studies $I^{\boldsymbol{G}}_{\aleph_0}$ for certain graphs $\boldsymbol{G}$ invariant under an action of a countable group and charaterizes when a set A belongs in $I^{\boldsymbol{G}}_{\aleph_0}$ in terms of dichotomies concerning homomorphisms of certain graphs to $\boldsymbol{G}|A$.

4.6 Hedetniemi's Conjecture

The *product* of two graphs $\boldsymbol{G} = (X, G), \boldsymbol{H} = (Y, H)$ is the graph $\boldsymbol{G} \times \boldsymbol{H} = (X \times Y, G \times H)$, with $(x, y) G \times H (x', y') \iff xGx' \;\&\; yHy'$. Clearly $\chi(\boldsymbol{G} \times \boldsymbol{H}) \leq \min\{\chi(\boldsymbol{G}), \chi(\boldsymbol{H})\}$. A well-known conjecture of Hedetniemi states that for any two finite graphs $\boldsymbol{G}$ and $\boldsymbol{H}$,

$$\chi(\boldsymbol{G} \times \boldsymbol{H}) = \min\{\chi(\boldsymbol{G}), \chi(\boldsymbol{H})\}.$$

See Zhu [Z] for a survey of Hedetniemi's Conjecture. Let $C(k)$, for $k \geq 2$, be the following statement: For any finite graphs $\boldsymbol{G}, \boldsymbol{H}$,

$$\chi(\boldsymbol{G}), \chi(\boldsymbol{H}) \geq k \implies \chi(\boldsymbol{G} \times \boldsymbol{H}) \geq k.$$

Then Hedetniemi's Conjecture is equivalent to $\forall k \geq 2\ C(k)$. It is obvious that $C(2)$ holds and it is easy to see that $C(3)$ holds, since the product of two odd cycles contains an odd cycle. El-Zahar and Sauer [ES] proved $C(4)$ but $C(k)$ for $k \geq 5$ is unknown. The version of Hedetniemi's Conjecture for infinite graphs is false. Hajnal [H85] constructed two graphs $\boldsymbol{G}, \boldsymbol{H}$ with $\chi(\boldsymbol{G}) = \chi(\boldsymbol{H}) = \aleph_1$ but $\chi(\boldsymbol{G} \times \boldsymbol{H}) = \aleph_0$; see also [KST, §6,(E),(F)], [R]. However we can ask if Hedetniemi's Conjecture holds for Borel chromatic numbers.

Problem 4.23. *If $\boldsymbol{G}, \boldsymbol{H}$ are analytic graphs, is it true that*

$$\chi_B(\boldsymbol{G} \times \boldsymbol{H}) = \min\{\chi_B(\boldsymbol{G}), \chi_B(\boldsymbol{H})\}?$$

For each $k \in \{2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$, let $C_B(k)$, the following statement: For any analytic graphs $\boldsymbol{G}, \boldsymbol{H}$,

$$\chi_B(\boldsymbol{G}), \chi_B(\boldsymbol{H}) \geq k \implies \chi_B(\boldsymbol{G} \times \boldsymbol{H}) \geq k.$$

Equivalently, Problem 4.23 can be stated as follows: Is the statement $C_B(k)$ true for all $k \in \{2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$? Trivially $C_B(2)$ holds. An immediate consequence of the $\boldsymbol{G}_0$-dichotomy gives an affirmative answer when $k = 2^{\aleph_0}$.

Corollary 4.24 ([KST, 6.11])**.** *If $\boldsymbol{G}, \boldsymbol{H}$ are analytic graphs with uncountable Borel chromatic number, then $\boldsymbol{G} \times \boldsymbol{H}$ also has uncountable Borel chromatic number.*

Using Corollary 4.24 and a result of Louveau [Lo] (see Theorem 4.31 below), one can also establish $C_B(3)$.

Proposition 4.25. *If $\boldsymbol{G}, \boldsymbol{H}$ are analytic graphs with $\chi_B(\boldsymbol{G}), \chi_B(\boldsymbol{H}) \geq 3$, then $\chi_B(\boldsymbol{G} \times \boldsymbol{H}) \geq 3$.*

Proof. First notice that we can assume that $\boldsymbol{G} = (X, G), \boldsymbol{H} = (Y, H)$ are bipartite. Because if $y_0, y_1, \ldots, y_{2n+1} = y_0$ is a simple odd cycle of length $2n+1, n > 0$, contained in $\boldsymbol{H}$, and c is a Borel 2-coloring of $\boldsymbol{G} \times \boldsymbol{H}$, then $c'(x) = c(x, y_0)$ is a Borel 2-coloring of $\boldsymbol{G}$.

Therefore, by Theorem 4.31, $\chi_B(\boldsymbol{G}^{odd}), \chi_B(\boldsymbol{H}^{odd})$ are uncountable, so by Corollary 4.24, $\chi_B(\boldsymbol{G}^{odd} \times \boldsymbol{H}^{odd})$ is uncountable, and, since $\boldsymbol{G}^{odd} \times \boldsymbol{H}^{odd}$ is a subgraph of $(\boldsymbol{G} \times \boldsymbol{H})^{odd}$, it follows that the Borel chromatic number of $(\boldsymbol{G} \times \boldsymbol{H})^{odd}$ is uncountable. The graph $\boldsymbol{G} \times \boldsymbol{H}$ is also bipartite, so, by Theorem 4.31 again, $\chi_B(\boldsymbol{G} \times \boldsymbol{H}) \geq 3$. $\square$

We do not know any other results along these lines. Is $C_B(4)$ true? Also in [H85] it is shown that (for arbitrary graphs) if $\chi(\boldsymbol{G}) \geq \aleph_0, \chi(\boldsymbol{H}) < \aleph_0$, then $\chi(\boldsymbol{G} \times \boldsymbol{H}) = \chi(\boldsymbol{H})$. Is this true for Borel chromatic numbers of analytic graphs?

Finally in [TV, Proposition 5.2] a counterexample to a 'lightface" version of Problem 4.23 is given. It is shown that there exist two (lightface) Δ^1_1 graphs $\boldsymbol{G}, \boldsymbol{H}$ which have no finite Δ^1_1 coloring but the product $\boldsymbol{G} \times \boldsymbol{H}$ has a Δ^1_1 3-coloring.

Addendum. It has been now shown in [Sh] that Hedetniemi's Conjecture fails: There is a large k such that $C(k)$ is false. However it is not clear what happens for small k, for example $C(5)$ is still open.

4.7 Extensions

As pointed out by Louveau [Lo], the proof of the $\boldsymbol{G}_0$-dichotomy theorem in [KST] actually shows the following stronger statement about directed graphs. Let $\boldsymbol{D}_0$ be the directed graph $\boldsymbol{D}_0^{(s_n)} = \boldsymbol{D}_0 = (2^{\mathbb{N}}, D_0)$ where

$$xD_0y \iff \exists n(x|n = y|n = s_n \;\&\; x(n) < y(n) \;\&\; \forall m > n(x(m) = y(m)).$$

Clearly $\boldsymbol{G}_{\boldsymbol{D}_0} = \boldsymbol{G}_0$. Then we have that for any analytic directed graph $\boldsymbol{D} = (X, D)$ on a Polish space X, exactly one of the following holds:

(i) $\chi_B(\boldsymbol{D}) \leq \aleph_0$,

(ii) $\boldsymbol{D}_0 \preceq_c \boldsymbol{D}$.

Furthermore, Lecomte [L09] and Louveau [Lo] also pointed out that the $\boldsymbol{G}_0$-dichotomy extends to finite-dimensional directed hypergraphs. The papers Lecomte [L09] and Miller [M11], [M12] contain appropriate generalizations of the $\boldsymbol{G}_0$-dichotomy to infinite-dimensional directed hypergraphs.

Certain extensions of the $\boldsymbol{G}_0$-dichotomy to κ-Souslin graphs are proved in [Ka]. Also extensions of the $\boldsymbol{G}_0$-dichotomy for graphs and hypergraphs in the context of determinacy are given in [CaK]

If X is a Hausdorff topological space, then $(X)^2$ is an open subspace of X^2. Note that the graph $\boldsymbol{G}_0$ is a (relatively) closed subset of $(2^{\mathbb{N}})^2$. For open graphs (even in analytic spaces), Feng proved the following stronger dichotomy, which can be viewed as a definable version of the *Open Coloring Axiom* (OCA) of Todorcevic, see [To].

Theorem 4.26 ([Fe, 1.1]). *Let X be a Polish space and $A \subseteq X$ an analytic subset of X. If $\boldsymbol{G} = (X, G)$ is a graph such that $G \subseteq A^2$ and G is open in A^2, then exactly one of the following holds:*

(i) $\chi_B(\boldsymbol{G}) \leq \aleph_0$,

(ii) $K_{2^{\mathbb{N}}} \preceq_c G$ *(or equivalently that is a Cantor set $C \subseteq X$ which is a $\boldsymbol{G}$-clique, i.e., $(C)^2 \subseteq G$).*

Remark 4.27. For $\boldsymbol{G}$ as in Theorem 4.26 it follows that $\chi_B(\boldsymbol{G}) \leq \aleph_0$ iff $\chi(\boldsymbol{G}) \leq \aleph_0$. Answering a question in an earlier version of this paper, B. Miller, D. Lecomte and S. Todorcevic (private communication) have pointed out that if G is open in A^2, then $\chi(\boldsymbol{G}) = \chi_B(\boldsymbol{G})$. This is because in an open graph, the closure of an independent set is independent and so given any classical coloring with countably many colors, one can find a $\boldsymbol{\Delta}^0_2$-measurable (and hence Borel) coloring with the same number of colors.

In the papers Frick-Geschke [FG], Geschke [G13], the authors study clopen (in $(X)^2$) graphs and hypergraphs.

4.8 Some dichotomy theorems of Louveau

In Louveau [Lo] the author studies classes $\mathcal{C}$ of directed analytic graphs on Polish spaces omitting certain cycles. He proves, among other results, dichotomy theorems concerning the question of whether a given directed graph $\boldsymbol{D}$ in $\mathcal{C}$ is $\preceq_B$ a countable (i.e., having countable vertex set) member of $\mathcal{C}$ and discusses how this is related to the countability of the Borel chromatic number of certain directed graphs associated to $\boldsymbol{D}$.

For example, note that for an arbitrary analytic directed graph $\boldsymbol{D}$ on a Polish space the following are equivalent: (a) $\chi_B(\boldsymbol{D}) \leq \aleph_0$ and (b) there is countable directed graph $\boldsymbol{C}$ such that $\boldsymbol{D} \preceq_B \boldsymbol{C}$. So, by (the directed version of) the $\boldsymbol{G}_0$-dichotomy, we have that for any analytic directed graph $\boldsymbol{D}$ on a Polish space X exactly one of the following holds: (i) There is countable directed graph $\boldsymbol{C}$ with $\boldsymbol{D} \preceq_B \boldsymbol{C}$ or (ii) $\boldsymbol{D}_0 \preceq_c \boldsymbol{D}$. In particular, if $\boldsymbol{D}$ is an analytic directed graph and $\boldsymbol{D} \preceq_B \boldsymbol{E}$, where $\boldsymbol{E}$ is a locally countable analytic directed graph, then for each connected component C of $\boldsymbol{D}$, we have that $\chi_B(\boldsymbol{D}|C) \leq \aleph_0$. Day and Marks [DM] have answered a question posed in an earlier version of this paper by showing that the converse of this is false. In fact, there is a undirected Borel graph $\boldsymbol{G}$ such that for each connected component C of $\boldsymbol{G}$, $\chi_B(\boldsymbol{G}|C) \leq \aleph_0$, however, there is no Borel homomorphism from $\boldsymbol{G}$ to a locally countable Borel graph. It would be also interesting to characterize those analytic directed graphs $\boldsymbol{D}$ for which $\boldsymbol{D} \preceq_B \boldsymbol{E}$, for a *locally finite* analytic directed graph $\boldsymbol{E}$.

For each $n \geq 2$, Louveau [Lo] defines a directed graph $\boldsymbol{D}_{0,n}$ (actually denoted by $\boldsymbol{G}_{0,n}$ in [Lo, §3]), related to $\boldsymbol{D}_0$ (in some sense $\boldsymbol{D}_{0,n}$ is the "nth root" of $\boldsymbol{D}_0$), such that $\boldsymbol{D}_{0,n} \not\preceq_B \boldsymbol{C}$, for any countable directed graph $\boldsymbol{C}$. For a binary relation $R \subseteq X^2$, let for $n \geq 2$, $R^{(n)}$ be the binary relation defined by

$$xR^{(n)}y \iff \exists x_0, \ldots, x_n(x_0 = x \,\&\, x_n = y \,\&\, \forall i < n(x_i R x_{i+1})).$$

Note that if $\boldsymbol{D} = (X, D)$ is a directed graph with no directed n-cycles, then $\boldsymbol{D}^{(n)} = (X, D^{(n)})$ is a directed graph. We now have:

Theorem 4.28 ([Lo, 3.2])**.** *Let $n \geq 2$ and let $\boldsymbol{D}$ be a directed analytic graph on a standard Borel space X. Then exactly one of the following holds:*

(i) $\chi_B(\boldsymbol{D}^{(n)} \cap (X)^2) \leq \aleph_0$,

(ii) $\boldsymbol{D}_{0,n} \preceq_c \boldsymbol{D}$.

Theorem 4.29 ([Lo, 4.6])**.** *Let $n \geq 2$ and let $\boldsymbol{D}$ be a directed analytic graph on a standard Borel space having no directed cycles of length $\leq n$. Then the following are equivalent:*

(a) $\boldsymbol{D} \preceq_B \boldsymbol{C}$, *for $\boldsymbol{C}$ a countable directed graph with no cycles of length $\leq n$,*

(b) $\forall k \leq n(\chi_B(\boldsymbol{D}^{(k)}) \leq \aleph_0)$.

Moreover, exactly one of the following holds:

(i) $\forall k \leq n(\chi_B(\boldsymbol{D}^{(k)}) \leq \aleph_0)$,

(ii) $\exists k \leq n(\boldsymbol{D}_{0,k} \preceq_B \boldsymbol{D})$.

An analogous result is proved in [Lo, 4.9] for analytic graphs having no odd cycles of length $\leq n$, with $n \geq 3$, replacing the graphs $\boldsymbol{D}_{0,k}$ by their symmetrizations $\boldsymbol{G}_{\boldsymbol{D}_{0,k}}$, and restricting the k in Theorem 4.29 to odd numbers.

Next consider the class of all directed analytic graphs with no directed cycles. For a binary relation R, let $R^* = \bigcup_n R^{(n)}$, so that if $\boldsymbol{G} = (X, G)$ is a graph, then $E_{\boldsymbol{G}} = G^*$. For a directed graph $\boldsymbol{D} = (X, D)$, let $\boldsymbol{D}^* = (X, D^*)$. If $\boldsymbol{D}$ has no directed cycles, then $\boldsymbol{D}^*$ is a (strict) poset. Thus a directed graph $\boldsymbol{D}$ with no directed cycles is $\preceq_B$ a countable directed graph with no directed cycles iff $\boldsymbol{D} \preceq_B \langle \mathbb{Q}, < \rangle$. Louveau [Lo] defines a continuum-size family of directed Borel graphs $\boldsymbol{D}_{0,\theta}$ (again denoted by $\boldsymbol{G}_{0,\theta}$ in [Lo, §4]), for $\theta \colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$, generalizing the graphs $\boldsymbol{D}_{0,n}$ (which correspond to $\theta \equiv n$) such that $\boldsymbol{D}_{0,\theta} \not\preceq_B \langle \mathbb{Q}, < \rangle$ and proves the following dichotomy theorem:

Theorem 4.30 ([Lo, 4.7, 4.12]). *Let $\boldsymbol{D}$ be a directed analytic graph with no directed cycles. Then the following are equivalent:*

(a) $\boldsymbol{D} \preceq_B \langle \mathbb{Q}, < \rangle$,

(b) $\chi_B(\boldsymbol{D}^*) \leq \aleph_0$.

Moreover exactly one of the following holds:

(i) $\chi_B(\boldsymbol{D}^*) \leq \aleph_0$,

(ii) $\exists \theta (\boldsymbol{D}_{0,\theta}) \preceq_B \boldsymbol{D})$.

Finally consider bipartite graphs $\boldsymbol{G} = (X, G)$. Recall that $\boldsymbol{G}$ is bipartite iff $\chi(\boldsymbol{G}) \leq 2$ iff $\boldsymbol{G}$ has no odd cycles. In this case the equivalence relation $E_{\boldsymbol{G}}$ contains the subequivalence relation $E_{\boldsymbol{G}}^{\text{even}}$, where $x E_{\boldsymbol{G}}^{\text{even}} y$ iff the $\boldsymbol{G}$-distance between x, y is even, so every $E_{\boldsymbol{G}}$-class contains at most two $E_{\boldsymbol{G}}^{\text{even}}$-classes. Note that for bipartite analytic $\boldsymbol{G}$ we have that $\chi_B(\boldsymbol{G}) \leq 2$ iff there is a Borel set that meets each non-singleton $E_{\boldsymbol{G}}$-class in exactly one $E_{\boldsymbol{G}}^{\text{even}}$-class. For $\theta \colon \mathbb{N} \to \mathbb{N}$, let $\boldsymbol{H}_{0,2\theta-1} = \boldsymbol{G}_{\boldsymbol{D}_{0,2\theta-1}}$, a bipartite graph. Then it turns out that $\chi_B(\boldsymbol{H}_{0,2\theta-1}) \geq 3$ and Louveau shows the following result, where for a bipartite graph $\boldsymbol{G} = (X, G)$, we define the graph $\boldsymbol{G}^{odd} = (X, G^{odd})$, by

$$x G^{odd} y \text{ iff there is a } \boldsymbol{G}\text{-path of odd length from } x \text{ to } y.$$

Theorem 4.31 ([Lo, 4.10, 4.13]). *Let $\boldsymbol{G} = (X, G)$ be a bipartite analytic graph. Then the following are equivalent:*

(a) $\chi_B(\boldsymbol{G}) \leq 2$,

(b) $\chi_B(\boldsymbol{G}^{odd}) \leq 2$,

(c) $\chi_B(\boldsymbol{G}^{odd}) \leq \aleph_0$.

Moreover, exactly one of the following holds:

(i) $\chi_B(\boldsymbol{G}) \leq 2$,

(ii) $\exists\theta(H_{0,2\theta-1} \preceq_B \boldsymbol{G})$.

A strengthening of this result can be found in Miller [Mu3, §4].

In [CMT-D, Proposition A.1] the authors prove the following, which they point out can be derived from results in [Lo] but they also prove directly using Theorem 4.33 below.

Theorem 4.32 ([CMT-D])**.** *Let $\boldsymbol{G}$ be a locally countable Borel graph on a standard Borel space X. Then the following are equivalent:*

(i) $\chi_B(\boldsymbol{G}) \leq 2$,

(ii) $\chi_M(\boldsymbol{G}) \leq 2$,

(iii) $\chi_{BM}(\boldsymbol{G}) \leq 2$, *for every Polish topology generating the Borel structure of X.*

4.9 Embedding $\boldsymbol{G}_0$

The following is a strengthening of the $\boldsymbol{G}_0$-dichotomy for certain analytic graphs.

Theorem 4.33 ([KST, 6.6])**.** *Let X be a Polish space and $\boldsymbol{G} = (X, G)$ an analytic graph which is either acyclic or locally countable. Then exactly one of the following holds:*

(i) $\chi_B(\boldsymbol{G}) \leq \aleph_0$,

(ii) $\boldsymbol{G}_0 \preceq_c^{\mathrm{inj}} \boldsymbol{G}$.

In fact in [KST, 6.6], this is proved under the weaker hypothesis that $\boldsymbol{G}$ is "almost acyclic". Also a similar result holds for directed graphs $\boldsymbol{D}$ replacing $\boldsymbol{G}_0$ by $\boldsymbol{D}_0$.

Lecomte-Miller [LM, Theorem 15] strengthened a special case of Theorem 4.33 as follows:

Theorem 4.34 ([LM, Theorem 15]). *Let X be a Polish space and $\boldsymbol{G} = (X, G)$ an analytic graph which is acyclic and locally countable. Then exactly one of the following holds:*

(i) $\chi_B(\boldsymbol{G}) \leq \aleph_0$,

(ii) $\boldsymbol{G}_0 \sqsubseteq_c \boldsymbol{G}$.

In particular, for any dense sequences $(s_n), (t_n), \boldsymbol{G}_0^{(s_n)}$ is homeomorphic to an induced subgraph of $\boldsymbol{G}_0^{(t_n)}$. However Miller [M08, paragraph preceding Theorem 3.3], showed that $\boldsymbol{G}_0^{(s_n)}, \boldsymbol{G}_0^{(t_n)}$ might not be Borel isomorphic. Another continuous embedding dichotomy theorem for having countable Borel chromatic number for certain kinds of directed graphs is given in Miller [Mu1]. A different extension of Theorem 4.34 is contained in [L16, Theorem 1.8]

Concerning the hypotheses about $\boldsymbol{G}$ in Theorem 4.34, it follows from Theorem 4.40 below that the conclusion fails if we only assume that $\boldsymbol{G}$ is locally countable. However the following is open:

Problem 4.35. *Does Theorem 4.34 go through if we only assume that the analytic graph $\boldsymbol{G}$ is acyclic?*

Remark 4.36. In the Borel case of Theorem 4.34 the graph is potentially F_σ. Lecomte (private communication) mentioned that he recently generalized this to other classes of graphs, which include graphs of arbitrarily high potential Borel complexity.

In [KST, §6 (A)], it was conjectured that Theorem 4.33 would generalize to all analytic graphs. However Lecomte showed that this conjecture fails. In fact he shows the following:

Theorem 4.37 ([L07, Theorem 10]). *There is no analytic graph $\boldsymbol{H}_0 = (X_0, H_0)$ on a standard Borel space X_0 such that for any Borel graph $\boldsymbol{G} = (X, G)$ on a standard Borel space X exactly one of the following holds:*

(i) $\chi_B(\boldsymbol{G}) \leq \aleph_0$,

(ii) $\boldsymbol{H}_0 \preceq_B^{inj} \boldsymbol{G}$.

In particular, there is a Borel graph $\boldsymbol{G}$ on a Polish space X such that $\chi_B(\boldsymbol{G}) = 2^{\aleph_0}$ but it is not the case that $\boldsymbol{G}_0 \preceq_{BM}^{inj} \boldsymbol{G}$.

In fact Theorem 4.37 is proved in [L07] by showing the following, interesting in its own right, stronger result:

Theorem 4.38 ([L07, §3])**.** *There is a Borel graph $\boldsymbol{G}$ (in fact of the form $\boldsymbol{G} = \boldsymbol{G}_{(F_n)}$, for a countable sequence of Borel functions (F_n)) which has uncountable Borel chromatic number but such that every locally countable Borel subgraph of $\boldsymbol{G}$ has countable Borel chromatic number.*

On the other hand, notice that Theorem 4.33 shows that for every analytic *acyclic* graph $\boldsymbol{G}$, $\boldsymbol{G}$ has uncountable Borel chromatic number iff some locally countable Borel subgraph of $\boldsymbol{G}$ has uncountable Borel chromatic number.

4.10 Basis and antibasis theorems

A *quasi-order* is a structure $\boldsymbol{Q} = (X, Q)$, where $Q \subseteq X^2$, and Q is transitive and reflexive. A *minimal element* of Q is an element $x \in X$ such that $yQx \implies xQy$. A *basis* for $\boldsymbol{Q}$ is a subset $B \subseteq X$ such that $\forall x \exists y \in B(yQx)$. A $\boldsymbol{Q}$*-antichain* is a subset $A \subseteq X$ such that if $x \neq y$ are in A, then it is not the case that xQy.

The $\boldsymbol{G}_0$-dichotomy tells us that $\{\boldsymbol{G}_0\}$ is a one-element basis for any quasi-order $\boldsymbol{Q}$ such that $\preceq_c \subseteq Q$ on the class of analytic graphs with uncountable Borel chromatic number (and similarly for $\boldsymbol{D}_0$). On the other hand, Theorem 4.37 shows that there is no one-element basis for $\preceq_B^{inj}$ in the class of analytic graphs of uncountable Borel chromatic number. The following is open:

Problem 4.39. *Is there a basis of cardinality $< 2^{\aleph_0}$ for $\preceq_B^{inj}$ among analytic graphs of uncountable Borel chromatic number? If this is not the case, is there such a basis consisting of a continuum size family of "reasonably simple" graphs.*

In the paper [LZ4] the authors study analytic directed graphs of uncountable Borel chromatic number and, among other results, show that any basis for $\preceq_B^{inj}$ among such directed graphs must be infinite.

In the series of papers Lecomte [L05], Lecomte [L07], Lecomte-Miller [LM], the authors study the basis problem on analytic or Borel graphs of uncountable Borel chromatic number for quasi-orders $\boldsymbol{Q}$ such that $\sqsubseteq_c \subseteq Q \subseteq \leq_{GB}$, i.e., for various reducibility (as opposed to homomorphism) notions. We state below some of their results that show, in particular, that every $\boldsymbol{Q}$-basis (for $\boldsymbol{Q}$ as above) for the analytic graphs of uncountable Borel chromatic number has maximum cardinality.

Theorem 4.40 ([L05, Theorem 8], [L07, Theorem 6]; see also [LM, Theorem 11])**.** *There are $2^{\aleph_0}$ locally countable Borel graphs of uncountable Borel chromatic number which are minimal and pairwise incomparable for any quasi-order $\boldsymbol{Q}$ with $\sqsubseteq_c \subseteq Q \subseteq \leq_{GB}$ among Borel graphs of uncountable Borel chromatic number. In particular, every $\boldsymbol{Q}$-basis for the Borel (or analytic) graphs of uncountable Borel chromatic number has size $2^{\aleph_0}$.*

Theorem 4.41 ([LM, Theorem 20])**.** *Let $\boldsymbol{Q}$ be any quasi-order among Borel (or analytic) graphs of uncountable Borel chromatic number such that $\sqsubseteq_c \subseteq Q \subseteq \leq_{GB}$. Then no $\boldsymbol{Q}$-antichain forms a $\boldsymbol{Q}$-basis.*

On the other hand, Lecomte and B. Miller show that there is a continuum size family of reasonably simple Borel directed graphs, somewhat reminiscent of $\boldsymbol{D}_0$, which form a $\sqsubseteq_c$-basis for the analytic locally countable directed graphs of uncountable Borel chromatic number, see [LM, Theorem 14].

Consider pairs of the form $S = (S^0, S^1)$, where $S^k \subseteq \bigcup_n (2^n \times 2^n)$, for $k = 0, 1$. For any such S, define the directed graph $\boldsymbol{D}^S = (2^{\mathbb{N}}, D^S)$ as follows, where for $k = 0, 1$, we let $\bar{k} = 1 - k$:

$$D^S = \{(s\hat{\ }k\hat{\ }\alpha, t\hat{\ }\bar{k}\hat{\ }\alpha) \colon k \in \{0,1\} \ \&\ (s,t) \in S^k \ \&\ \alpha \in 2^{\mathbb{N}}\}.$$

Call S *dense* if $\forall r \in 2^{<\mathbb{N}} \exists (s,t) \in S^0 (r \subseteq s, t)$. Then we have:

Theorem 4.42 ([LM, Theorem 14])**.** *Let X be a Polish space and $\boldsymbol{D} = (X, D)$ an analytic directed graph which is locally countable. Then exactly one of the following holds:*

(i) $\chi_B(\boldsymbol{D}) \leq \aleph_0$,

(ii) There is dense S such that $\boldsymbol{D}^S \sqsubseteq_c \boldsymbol{D}$.

Theorem 4.31 shows that there is a basis for $\preceq_B$ among bipartite analytic graphs with $\chi_B \geq 3$, which consists of continuum many explicitly given such graphs. It is unknown however whether this is optimal, in the sense that any such basis must have the cardinality of the continuum. On the other hand, Miller [Mu3, Theorem 23] shows that every basis for $\preceq_B$ on the bipartite analytic *directed* graphs with $\chi_B \geq 3$ must have continuum size. By similar methods, the same conclusion can be derived for $\preceq_B^{inj}$ among bipartite analytic graphs with $\chi_B \geq 3$.

For an analytic graph $\boldsymbol{G}$ on a standard Borel space X, consider the supremum of $\chi_{BM}(\boldsymbol{G})$ over all the Polish topologies that generate its Borel structure. Then from the $\boldsymbol{G}_0$-dichotomy it follows that $\{\boldsymbol{G}_0\}$ is a basis for $\preceq_B$ among all analytic graphs $\boldsymbol{G}$ for which this supremum is uncountable. On the other hand, we do not know if there is a reasonable basis under $\preceq_B$ among all analytic graphs $\boldsymbol{G}$ for which $\chi_M(\boldsymbol{G})$ is uncountable. In fact we do not know if there is even a one-element basis, i.e., the following is open:

Problem 4.43. *Is there an analytic graph $\boldsymbol{G}_M$ such that for any analytic graph $\boldsymbol{G}$ exactly one of the following holds:*

(i) $\chi_M(\boldsymbol{D}) \leq \aleph_0$,

(ii) $\boldsymbol{G}_M \preceq_B \boldsymbol{G}$.

4.11 Refinements

Lecomte and Zeleny [LZ1], [LZ2], [LZ3] consider the Borel complexity of countable colorings. In [LZ2] they show that for each $\xi \geq 1$, there is a Borel (in fact a difference of two closed sets) graph with Borel chromatic number 2 but which has no $\boldsymbol{\Delta}^0_\xi$-measurable countable coloring. In [LZ3] they show that for each $\xi \geq 1$, there is a closed graph with Borel chromatic number 2 and $\boldsymbol{\Delta}^0_\xi$-measurable chromatic number $\aleph_0$.

In [LZ1] they propose a dichotomy conjecture (analogous to the $\boldsymbol{G}_0$-dichotomy) to characterize when an analytic directed graph admits a $\boldsymbol{\Delta}^0_\xi$-measurable countable coloring and prove this conjecture for $\xi \leq 3$. A different approach and a more canonical version, as well as a generalization, of such a dichotomy for $\xi = 2$ is contained in [CMS, Theorem 4.4]

Finally some connections with the $\boldsymbol{G}_0$-dichotomy are discussed in the recent memoir by Lecomte [L13], which proves a general Hurewicz-like dichotomy for characterizing when Borel relations are in a given potentially Wadge class (and in fact a related separation result). Dichotomy results concerning potential Wadge classes of *oriented* (i.e., antisymmetric, directed) graphs are also proved in [Lo]. Lecomte (private communication) pointed out that these concern the classes $\boldsymbol{\Delta}^0_1, \boldsymbol{\Delta}^0_2$, while recently Zamora, using results in [L13], generalized them to all Borel Wadge classes and also all $\boldsymbol{\Delta}^0_\xi$. Further basis and antibasis results concerning certain classes of graphs and other relations related to potential Wadge classes are included in [L16, Theorem 1.9, Corollary 1.10 and Theorem 1.11]. See also [LZ].

4.12 B. Miller's graph theoretic approach to dichotomy theorems

An important role in descriptive set theory is played by dichotomy theorems that sharply delineate the boundary between simple and complicated structure. A classical example is Souslin's Perfect Set Theorem for analytic sets: If A is an analytic set in a Polish space, then exactly one of the following holds: (i) A is countable or (ii) A contains a Cantor set. More recently dichotomy theorems have been prominent in the theory of definable equivalence relations. Silver's Theorem [Si] states that for any coanalytic equivalence relation E on a Polish space X exactly one of the following holds: (i) E has countably many classes or (ii) there is a Cantor set of E-inequivalent elements. The General Glimm-Effros Dichotomy, see [HKL], states that for every Borel equivalence relation E on a Polish space X exactly one of the following holds: (i) E is *smooth*, i.e., E is Borel reducible to equality on some Polish space Y or (ii) $E_0 \sqsubseteq_c E$. An extensive survey of such dichotomy results for definable equivalence relations is contained in [HK97], [HK01]. An interesting feature of the "modern" dichotomy theorems, including the Silver, General Glimm-Effros, $\boldsymbol{G}_0$, etc., dichotomies, has been that their proofs used "non-classical" techniques, especially forcing and effective descriptive set theory. It had been an open problem whether such results could be proved by classical techniques.

In an important development over the last few years, B. Miller has found a new unified approach to many dichotomy theorems in descriptive set theory, based on graph theoretic dichotomies extending the $\boldsymbol{G}_0$-dichotomy. This theory achieves the following goals: (i) Provides proofs based on classical methods, like Cantor-Bendixson type derivatives, Baire category arguments, etc., for many dichotomy theorems, whose only earlier known proofs used "non-classical" methods; (ii) Gives a unified picture encompassing many old and new descriptive dichotomy theorems, which are now derived from appropriate graph (or hypergraph) dichotomies; (iii) Naturally adapts to the context of more general (than Polish) spaces or to more extensive notions of definability like, e.g., κ-Souslin, for which some of the earlier proofs did not work.

As a simple illustration, let us see, for example, how one can derive the Silver Dichotomy from the $\boldsymbol{G}_0$-dichotomy. Let E be a coanalytic equivalence relation on a Polish space X and consider the analytic graph $\boldsymbol{G} = (X, G)$, where $G = X^2 \setminus E$. If $\boldsymbol{G}$ has countable Borel chromatic number, then clearly E has countably many classes, since any $\boldsymbol{G}$-independent set is contained in a single E-class. Otherwise, by the $\boldsymbol{G}_0$-dichotomy, there is a continuous function $f\colon 2^{\mathbb{N}} \to X$ such that $xG_0y \implies \neg f(x)Ef(y)$. Let F be the analytic equivalence relation on $2^{\mathbb{N}}$ which is the preimage of E by f: $xFy \iff f(x)Ef(y)$. Then every F-class C is meager, since if it is not, then by Example 4.16 it cannot be $\boldsymbol{G}_0$-independent. So there are $x, y \in C$ with xG_0y and also $f(x)Ef(y)$, a contradiction. Thus, by the Kuratowski-Ulam Theorem, F itself is meager and so by the Kuratowski-Mycielski Theorem (see [K95, 19.1]) there is a Cantor set of F-inequivalent elements. Its image under f is then a Cantor set of E-inequivalent elements.

An introduction to Miller's theory is contained in [M12] and a more detailed development can be found in the series of lecture notes [Mu2] and in [M11].

5. Finite vs. countably infinite Borel chromatic numbers

5.1 Examples of graphs with countably infinite Borel chromatic number

We discuss here a couple of interesting examples of such graphs, whose calculation of the Borel chromatic number is based on quite different techniques.

Example 5.1. The *odd distance graph* on $\mathbb{R}^2$ is the graph $\boldsymbol{O} = (\mathbb{R}^2, O)$, where xOy iff the distance between x, y is an odd integer. By Proposition 4.5 it is clear that $\chi_B(\boldsymbol{O}) \leq \aleph_0$. On the other hand, it is known that $\chi_\mu(\boldsymbol{O}) \geq \aleph_0$, where μ is a Lebesgue measure on $\mathbb{R}^2$, therefore $\chi_B(\boldsymbol{O}) = \chi_\mu(\boldsymbol{O}) = \aleph_0$. One way to prove that $\chi_\mu(\boldsymbol{O})$ is infinite is through the following result: If $A \subseteq \mathbb{R}^2$ is Lebesgue measurable and its upper density $\overline{d}(A) = \limsup_{R\to\infty} \frac{\mu(A\cap[-R,R]^2)}{4R^2}$ is positive, then for some D the set A contains pairs of points of every distance $\geq D$. See [Bo], [Bu], [FKW], [FM], [Q], [Mo] for proofs of this result, using various techniques from ergodic theory, harmonic analysis, geometric measure theory, or probability theory. The paper [Ste] presents another proof that $\chi_\mu(\boldsymbol{O})$ is infinite using spectral techniques. We do not know if also $\chi_{BM}(\boldsymbol{O}) = \aleph_0$. It is also unknown whether $\chi(\boldsymbol{O}) < \aleph_0$. It is shown in [AMRSS] that $\chi(\boldsymbol{O}) \geq 5$.

If instead of the odd distance graph one considers the graph on $\mathbb{R}^2$ where two points are connected by an edge iff their distance belongs to some prescribed unbounded (Borel) subset of the positive reals, then the above argument shows that it also has Borel and μ-measurable chromatic at least $\aleph_0$. By contrast the *unit distance graph* on $\mathbb{R}^2$, defined by $\boldsymbol{U} = (\mathbb{R}^2, U)$, where xUy iff the distance between x, y is equal to 1, is known to have $5 \leq \chi(\boldsymbol{U}) \leq \chi_\mu(\boldsymbol{U}) \leq \chi_B(\boldsymbol{U}) \leq 7$, see [DG] and [So], but the exact values of $\chi(\boldsymbol{U}), \chi_\mu(\boldsymbol{U}), \chi_B(\boldsymbol{U})$ are unknown. We refer to [So] for the long history of this problem for the unit distance graph and its higher-dimensional analogs and [BPT], [EI], and references contained therein, for more recent developments.

Example 5.2. Let $[\mathbb{N}]^{\mathbb{N}}$ be the set of strictly increasing sequences from $\mathbb{N}$. Let $S_{\mathbb{N}}$ be the shift map on $\mathbb{N}^{\mathbb{N}}, S_{\mathbb{N}}(p)(n) = p(n+1)$ and let $S_{\mathbb{N}}^{\infty}$ be the restriction of $S_{\mathbb{N}}$ to $[\mathbb{N}]^{\mathbb{N}}$. Then $\chi_B(\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}) = \aleph_0$. That $\chi_B(\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}) \leq \aleph_0$ follows from Corollary 4.6, while $\chi_B(\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}) \geq \aleph_0$ follows from the Galvin-Prikry Theorem [GP] which asserts that for every Borel partition $[\mathbb{N}]^{\mathbb{N}} = A_1 \sqcup \cdots \sqcup A_n$, there is $1 \leq i \leq n$ and infinite $H \subseteq \mathbb{N}$ with $[H]^{\mathbb{N}}(= H^{\mathbb{N}} \cap [\mathbb{N}]^{\mathbb{N}}) \subseteq A_i$. (On the other hand, by [M08], $\chi_{BM}(\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}) = 3$ and $\chi_M(\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}) = 3$ – see also Theorem 5.13 and Theorem 6.18 below.) For other related results, see [DPT2].

There is also a trivial way to construct such examples. Given a sequence $\boldsymbol{G}_n = (X_n, G_n)$ of graphs with pairwise disjoint X_n, their *direct sum* is the graph

$$\bigsqcup_n \boldsymbol{G}_n = (\bigcup_n X_n, \bigcup_n G_n).$$

Example 5.3. Take a sequence of Borel graphs $\boldsymbol{G}_n$ with $\chi_B(\boldsymbol{G}_n) < \infty$ but $\chi_B(\boldsymbol{G}_n) \to \infty$ and let $\boldsymbol{G} = \bigsqcup_n \boldsymbol{G}_n$. Then $\chi_B(\boldsymbol{G}) = \aleph_0$.

5.2 Bounded and locally finite degree graphs

Note that the graph in Example 5.2 is locally finite but of unbounded degree. For bounded degree graphs we have the following result.

Proposition 5.4 ([KST, 4.6])**.** *Let $\boldsymbol{G} = (X, G)$ be an analytic graph on a standard Borel space X. If $\Delta(\boldsymbol{G}) \leq d$, then $\chi_B(\boldsymbol{G}) \leq d+1$. Moreover, if $d \geq 1$, then $\chi'_B(\boldsymbol{G}) \leq 2d-1$, since $\Delta(\boldsymbol{L}(\boldsymbol{G})) \leq 2d-2$.*

Remark 5.5. Proposition 5.4 is proved in [KST, 4.6] for Borel graphs but easily extends to analytic graphs, since every analytic graph $\boldsymbol{G}$ with $\Delta(\boldsymbol{G}) \leq d$ is a subgraph of a Borel graph $\boldsymbol{H}$ with $\Delta(\boldsymbol{H}) \leq d$ and the same set of vertices. This follows from the First Reflection Theorem (see [K95, 35.10]).

By results of Marks in Section 5 below, see Theorems 6.3 and 6.12, these upper bounds are optimal, for d-regular, acyclic Borel $\boldsymbol{G}$.

On the other hand, we have the following result for analytic, locally finite graphs:

Theorem 5.6 ([CM2]). *Let $\boldsymbol{G} = (X, G)$ be a locally finite analytic graph on a Polish space X with $\chi(\boldsymbol{G}) < \infty$. Then $\chi_{BM}(\boldsymbol{G}) \leq 2\chi(\boldsymbol{G}) - 1$. Therefore, if $\boldsymbol{G}$ is bipartite (in particular acyclic), then $\chi_{BM}(\boldsymbol{G}) \leq 3$.*

The hypothesis of local finiteness in Theorem 5.6 cannot be dropped, as shown in Example 4.14. Also χ_{BM} in general cannot be replaced by χ_M, even in the acyclic case, see Theorem 6.59. It holds though for χ_M in the hyperfinite case, see Theorem 6.18.

It is not clear that the upper bound in Theorem 5.6 is optimal. The acyclic graph $\boldsymbol{G}$ generated by the free part of the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$ satisfies $\chi(\boldsymbol{G}) = 2$ but $\chi_{BM}(\boldsymbol{G}) = 3$, so $\chi_{BM}(\boldsymbol{G}) > \chi(\boldsymbol{G})$. However the following is open:

Problem 5.7. *Is there a bounded degree Borel graph $\boldsymbol{G}$ on a Polish space X for which $\chi_{BM}(\boldsymbol{G}) > \chi(\boldsymbol{G}) + 1$?*

5.3 Graphs generated by functions

As we have seen in Section 3, **(A)**, any locally countable Borel graph $\boldsymbol{G}$ is of the form $\boldsymbol{G} = \boldsymbol{G}_{(F_n)}$ for some countable family of Borel functions (F_n) and can have any Borel chromatic number. We will consider here the case of graphs $\boldsymbol{G} = \boldsymbol{G}_{F_1,\dots,F_n}$ generated by finitely many Borel functions. By Corollary 4.6, $\chi_B(\boldsymbol{G}_{F_1,\dots,F_n}) \leq \aleph_0$. What are the possible values of $\chi_B(\boldsymbol{G}_{F_1,\dots,F_n})$? In Example 5.2, we have a function F for which $\chi_B(\boldsymbol{G}_F) = \aleph_0$. The following problem was raised (in a somewhat more restricted form) in [KST, 4.9].

Problem 5.8. *Let $F_1, \dots, F_n$ be Borel functions on a standard Borel space X. Is it true that $\chi_B(\boldsymbol{G}_{F_1,\dots,F_n}) \in \{1, \dots, 2n+1, \aleph_0\}$?*

This is motivated by the standard result in graph theory which asserts that given a directed graph $\boldsymbol{D} = (X, D)$ with out-degree $\leq n$, we have $\chi(\boldsymbol{D}) \leq 2n + 1$ (see, e.g., [KST, p. 14]). Note that by a standard result in descriptive set theory (see [K95, 18.15]) the directed graphs of the form

$\boldsymbol{D}_{F_1,\dots,F_n}$, with $F_1,\dots F_n$ Borel, are exactly the Borel directed graphs with out-degree $\leq n$. Note also that if $F_1,\dots,F_n$ are functions on a set X and $F_i(x)\neq x, \forall i\leq n, x\in X$, then $\chi(\boldsymbol{G}_{F_1,\dots,F_n})\leq 2n+1$ iff there is a partition $X=A_1\sqcup\cdots\sqcup A_{2n+1}$ such that $F_i(A_j)\cap A_j=0, \forall i\leq n, j\leq 2n+1$. Similarly for its Borel version.

Finally note that the list $\{1,2,\dots,2n+1,\aleph_0\}$ in Problem 5.8 is irreducible, since for each $m\leq 2n+1$ there is a finite set X and $F_i\colon X\to X, 1\leq i\leq n$, with $\chi(\boldsymbol{G}_{F_1,\dots,F_n})=m$. For $m=2n+1$, let $X=\{0,\dots,2n\}$ and put $F_i(x)=(x+i) \mod (2n+1)$, for $1\leq i\leq n$. Then $\boldsymbol{G}_{F_1,\dots,F_n}$ is the complete graph on X. For any $1\leq m\leq 2n+1$, let X be finite, $F_i\colon X\to X, 1\leq i\leq n$, with $\chi(\boldsymbol{G}_{F_1,\dots,F_n})=2n+1$ and let $X=A_1\sqcup\cdots\sqcup A_{2n+1}$ be a partition in $2n+1$ non-empty independent sets. Let $Y=A_1\sqcup\cdots\sqcup A_m$ and define $H_i\colon Y\to Y, 1\leq i\leq n$, by $H_i(y)=y$, if $F_i(y)\notin Y$; else $H_i(y)=F_i(y)$. Clearly $\chi(\boldsymbol{G}_{H_1,\dots,H_n})\leq m$. If $\chi(\boldsymbol{G}_{H_1,\dots,H_n})<m$, with $Y=B_1\sqcup\cdots\sqcup B_\ell, \ell<m, B_i$ independent, $1\leq i\leq\ell$, then $X=B_1\sqcup\cdots\sqcup B_\ell\sqcup A_{m+1}\sqcup\cdots\sqcup A_{2n+1}$ shows that $\chi(\boldsymbol{G}_{F_1,\dots,F_n})<2n+1$, a contradiction.

Using the result of Marks [Ma1] that there is an acyclic graph of the form $\boldsymbol{G}_{F_1,\dots,F_n}, F_i$ Borel, $1\leq i\leq n$, with $\chi_B(\boldsymbol{G}_{F_1,\dots,F_n})=2n+1$, one can similarly find for each $1\leq m\leq 2n+1$, an acyclic graph of the form $\boldsymbol{G}_{H_1,\dots,H_n}, H_i$ Borel, $1\leq i\leq n$, with $\chi_B(\boldsymbol{G}_{H_1,\dots,H_n})=m$.

The following partial answers to Problem 5.8 have been obtained so far.

Theorem 5.9 ([KST, 5.1]). *For any Borel function F on a standard Borel space X, we have $\chi_B(\boldsymbol{G}_F)\in\{1,2,3,\aleph_0\}$.*

Remark 5.10. Although first explicitly stated and proved in that paper, as it is pointed out in [KST, p. 18], this result can be also derived (in a different way) from [KS, 2.2], by using the idea of changing topologies in a Polish space to make Borel functions continuous and Borel sets clopen. There have been also several other results in the literature along these lines concerning finite colorings of the graphs of a single continuous function, see, e.g., [KS], [vM] and references therein. Also note that Theorem 5.9 gives trivially that for any Borel functions $F_1,\dots,F_n$, $\chi_B(\boldsymbol{G}_{F_1,\dots,F_n})\in\{1,2,\dots,3^n,\aleph_0\}$.

We next have:

Theorem 5.11 ([Pa]). *For any commuting and fixed point free Borel functions $F_1,\dots,F_n$ on a standard Borel space, $\chi_B(\boldsymbol{G}_{F_1,\dots,F_n})\in\{1,2,\dots,2n+1,\aleph_0\}$.*

It is shown in [MP, Theorem 3.15,] that, for commuting Borel functions $F_1, \dots, F_n$ on a standard Borel space X with $\chi_B(\boldsymbol{G}_{F_1,\dots,F_n}) < \aleph_0$, if for each $x \in X$, there is a path from x to a fixed point of some F_i, then there is an increasing sequence of Borel sets X_m such that $\bigcup_m X_m = X$ and $\chi_B(\boldsymbol{G}_{F_1,\dots,F_n})|X_m \leq 2n$, $\forall m$. Moreover, in this case, for $n = 2$, one actually has $\chi_B(\boldsymbol{G}_{F_1,F_2}) \leq 4$, see [MP, Theorem 4.3], and, for any n, $\chi_M^{ap}(\boldsymbol{G}_{F_1,\dots,F_n}) \leq 2n+1$, see [MP, Corollary 3.22].

Theorem 5.12 ([Pa])**.** *(i) For Borel functions* F_1, F_2,

$$\chi_B(\boldsymbol{G}_{F_1,F_2}) \in \{1, 2, 3, 4, 5, \aleph_0\}.$$

(ii) For Borel functions F_1, F_2, F_3,

$$\chi_B(\boldsymbol{G}_{F_1,F_2,F_3}) \in \{1, 2, 3, 4, 5, 6, 7, 8, \aleph_0\}.$$

(iii) For Borel functions $F_1, F_2, \dots, F_n$,

$$\chi_B(\boldsymbol{G}_{F_1,\dots,F_n}) \in \{1, 2, \dots, \frac{1}{2}(n+1)(n+2), \aleph_0\}.$$

It is shown in [MP, Corollary 2.4] that $\frac{1}{2}(n+1)(n+2)$ can be improved to $\frac{1}{2}(n+1)(n+2) - 2$ in Theorem 5.12 (iii), when $n \geq 4$.

So Problem 5.8 has a positive answer for $n \leq 2$ and an almost positive answer for $n = 3$ (with 8 instead of the optimal 7). Note also that by Proposition 5.4, if $F_1, \dots, F_n$ are all $\leq k$-to-1, then $\chi_B(\boldsymbol{G}_{F_1,\dots,F_n}) \leq (k+1)n + 1$.

Concerning measure and Baire measurable chromatic numbers (on Polish spaces), we have the following results:

Theorem 5.13. *(i)* ([M08]) *For any Borel function* F, $\chi_{BM}(\boldsymbol{G}_F) \leq 3$, $\chi_M(\boldsymbol{G}_F) \leq 3$.

(ii) ([M08], [Pa]) *For Borel functions* F_1, F_2,

$$\chi_{BM}(\boldsymbol{G}_{F_1,F_2}) \leq 5, \chi_M(\boldsymbol{G}_{F_1,F_2}) \leq 5.$$

(iii) ([M08], [Pa]) *For Borel functions* F_1, F_2, F_3,

$$\chi_{BM}(\boldsymbol{G}_{F_1,F_2,F_3}) \leq 8, \chi_M(\boldsymbol{G}_{F_1,F_2,F_3}) \leq 8.$$

(iv) ([M08], [Pa]) *For Borel functions* $F_1, F_2, \dots, F_n$,

$$\chi_{BM}(\boldsymbol{G}_{F_1,\dots,F_n}) \leq \frac{1}{2}(n+1)(n+2), \chi_M(\boldsymbol{G}_{F_1,\dots,F_n}) \leq \frac{1}{2}(n+1)(n+2).$$

In view of Theorem 5.13 we have the following problem:

Problem 5.14. *Let $F_1, \dots, F_n$ be Borel functions on a Polish space X. Is it true that $\chi_{BM}(\boldsymbol{G}_{F_1,\dots,F_n}) \leq 2n+1$ and $\chi_M(\boldsymbol{G}_{F_1,\dots,F_n}) \leq 2n+1$?*

Also, by Theorem 5.6, we have the following partial answer to Problem 5.14 with a somewhat weaker, but still linear, upper bound.

Theorem 5.15 ([CM2])**.** *Let $F_1, \dots, F_n$ be Borel functions on a Polish space X such that the graph $\boldsymbol{G}_{F_1,\dots,F_n}$ is locally finite. Then $\chi_{BM}(\boldsymbol{G}_{F_1,\dots,F_n}) \leq 4n+1$.*

See also Theorem 6.18 for a similar upper bound for χ_M in the hyperfinite case.

Finally concerning directed graphs of the form $\boldsymbol{D}_F$, Miller [M08, §4, 5] showed that any basis under $\preceq_B$ for the class of the directed graphs $\boldsymbol{D}_F$, F Borel, with $\chi_B(\boldsymbol{D}_F) \geq 3$ must be quite complicated, in particular must have cardinality $2^{\aleph_0}$.

5.4 Some universality results

We note here a universality result concerning locally countable directed graphs originating in [KST, 7.1, 7.2]. It can be easily generalized to a universality result for structurable equivalence relations, see [ChK, Theorem 4.1].

Let $\mathcal{K}$ be a class of countable directed graphs closed under isomorphism. We say that $\mathcal{K}$ is *Borel* if the set of all directed graphs with vertex set contained in $\mathbb{N}$ that belong to $\mathcal{K}$ is a Borel set in the standard Borel space of all such directed graphs.

Examples of Borel classes $\mathcal{K}$ include the classes of countable (directed) graphs, acyclic countable (directed) graphs, locally finite (directed) graphs, bounded degree (directed) graphs, graphs of $\leq \aleph_0$-to-1, finite-to-1, $\leq k$-to-1 functions, k-chromatic graphs, etc.

A locally countable Borel directed graph $\boldsymbol{D} = (X, D)$ on a standard Borel space X is *locally in $\mathcal{K}$* if the restriction of $\boldsymbol{D}$ to every connected component of $\boldsymbol{G}_{\boldsymbol{D}}$ is in $\mathcal{K}$.

We now have the following universality result.

Proposition 5.16 ([KST, 7.1, 7.2])**.** *Let $\mathcal{K}$ be a Borel class of countable directed graphs closed under isomorphism. Then there is a locally countable Borel directed* *graph $\boldsymbol{D}^{\mathcal{K}}_\infty$ on a standard Borel space X_∞, such that $\boldsymbol{D}^{\mathcal{K}}_\infty$ is locally in $\mathcal{K}$ and for every locally countable Borel directed graph $\boldsymbol{D}$ which is locally in $\mathcal{K}$, there is a $E_{\boldsymbol{D}^{\mathcal{K}}_\infty}$-invariant Borel subset $Y \subseteq X_\infty$ such that $\boldsymbol{D} \cong_B \boldsymbol{D}^{\mathcal{K}}_\infty | Y$.*

For example, applying Proposition 5.16 to $\mathcal{K}$ = class of countable bipartite (acyclic) graphs, we obtain a universal bipartite (acyclic) locally countable Borel graph, which, by Example 4.16, has uncountable Borel chromatic number. On the other hand, if we fix a countable connected directed graph $\boldsymbol{E}$ and let $\mathcal{K}_{\boldsymbol{E}}$ be the class of all directed graphs isomorphic to $\boldsymbol{E}$, then we have the following problem:

Problem 5.17. *Calculate the Borel chromatic number of $\boldsymbol{D}^{\mathcal{K}_{\boldsymbol{E}}}_\infty$.*

In the particular case of graphs generated by functions, Proposition 5.16 takes the following form. For a given class $\mathcal{F}$ of Borel functions $F\colon X \to X$ on standard Borel spaces, we say that a function $F^{\mathcal{F}}_\infty\colon X_\infty \to X_\infty$ in $\mathcal{F}$ is *universal in $\mathcal{F}$* if for every $F\colon X \to X$ in $\mathcal{F}$ there is a Borel injection $\pi\colon X \to X_\infty$ with $\pi \circ F = F^{\mathcal{F}}_\infty \circ \pi$ and moreover $\pi(Y)$ is invariant under the equivalence relation generated by the graph of $F^{\mathcal{F}}_\infty$.

Corollary 5.18. *There is a universal Borel $\leq\aleph_0$-to-1 (resp., finite-to-1, $\leq k$-to-1) function F_∞ (resp., F^f_∞, F^k_∞).*

In particular, $\chi_B(\boldsymbol{G}_{F_\infty}) = \chi_B(\boldsymbol{G}_{F^f_\infty}) = \aleph_0$ and $\chi_B(\boldsymbol{G}_{F^k_\infty}) = 3$.

With a slightly weaker notion of universality, there is a concrete universal $\leq\aleph_0$-to-1 and $\leq k$-to-1 Borel function. Below for $n \in \{2, 3, \dots, \mathbb{N}\}$ we let $S_n\colon n^{\mathbb{N}} \to n^{\mathbb{N}}$ be the shift map $S_n(p)(i) = p(i+1)$.

Theorem 5.19 ([KST, 7.10])**.** *Let $F\colon X \to X$ be a $\leq k$-to-1 Borel function, where X is a standard Borel space, and $k = 2, 3, \dots$. Then there is a Borel map $\pi\colon X \to k^{\mathbb{N}}$ which is 1-1 on each $E_{\boldsymbol{G}_F}$-class and $\pi \circ F = S_k \circ \pi$. Similarly for any $\leq\aleph_0$-to-1 Borel function $F\colon X \to X$ and $S_{\mathbb{N}}$.*

Consider now the free semigroup $\mathbb{S}_n$ with n generators $\gamma_1, \dots, \gamma_n$. Let $S_{n,i}\colon \mathbb{N}^{\mathbb{S}_n} \to \mathbb{N}^{\mathbb{S}_n}, 1 \leq i \leq n$, be defined by $S_{n,i}(p)(g) = p(g\gamma_i)$. Thus if $n = 1, S_{1,1} = S_{\mathbb{N}}$. We now have the following universality result.

Theorem 5.20 ([KST, 7.7])**.** *Let $F_i\colon X \to X, 1 \leq i \leq n$, be $\leq\aleph_0$-to-1 Borel functions on a standard Borel space such that there are only countably many finite subsets of X closed under all $F_i, 1 \leq i \leq n$. Then there is a Borel injection $\pi\colon X \to \mathbb{N}^{\mathbb{S}_n}$ with $\pi \circ F_i = S_{n,i} \circ \pi, 1 \leq i \leq n$.*

In the case $n = 1$ this specializes to the following result, where for a function $F\colon X \to X$ a point $x \in X$ is *periodic* if the forward orbit, given by $\{F^{(n)}(x)\colon n \geq 0\}$, is finite.

Corollary 5.21 ([KST, 7.8])**.** *Let $F\colon X \to X$ be a $\leq\aleph_0$-to-1 Borel function on a standard Borel space X with only countably many periodic points. Then there is a Borel injection $\pi\colon X \to \mathbb{N}^{\mathbb{N}}$ with $\pi \circ F = S_{\mathbb{N}} \circ \pi$.*

There is no such analog for $\leq k$-to-1 functions, since there are aperiodic Borel automorphisms $F\colon X \to X$ on standard Borel spaces for which there is no Borel injection $\pi\colon F \to n^{\mathbb{N}}$ and $\pi \circ F = S_n \circ \pi$, for any finite n (see [W84]).

5.5 On the shift graph on $[\mathbb{N}]^{\mathbb{N}}$

Consider the graph $\boldsymbol{G}_F = (X, G_F)$ associated to a Borel $\leq\aleph_0$-to-1 function $F\colon X \to X$ on a standard Borel space X. Let

$$P^*(F) = \{x \in X : x \text{ is periodic}\}.$$

Then $P^*(F) \subseteq X$ is a Borel $E_{\boldsymbol{G}_F}$-invariant set which is *smooth*, i.e., there is a Borel set meeting each $E_{\boldsymbol{G}_F}$-class in $P^*(F)$ in a single point. From this it follows easily that $\boldsymbol{G}_F|P^*(F)$ has Borel chromatic number at most 3. By Corollary 5.21 there is a Borel injection $\pi\colon X \setminus P^*(F) \to \mathbb{N}^{\mathbb{N}}$ such that $\pi \circ F = \mathbb{S}_{\mathbb{N}} \circ \pi$ and thus if $Y = \pi(X \setminus P^*(F))$, the chromatic number of $\boldsymbol{G}_F$ is infinite iff the Borel chromatic number of $\boldsymbol{G}_{\mathbb{S}_{\mathbb{N}}}|Y$ is infinite. Next notice that $\boldsymbol{G}_{\mathbb{S}_{\mathbb{N}}}|(\mathbb{N}^{\mathbb{N}} \setminus [\mathbb{N}]^{\mathbb{N}})$ is finite (see [KST, pp. 37–38]). Thus $\chi_B(\boldsymbol{G}_F) = \aleph_0$ iff $\chi_B(\boldsymbol{G}_{\mathbb{S}_{\mathbb{N}}}|Y \cap [\mathbb{N}]^{\mathbb{N}}) = \chi_B(\boldsymbol{G}_{S^{\infty}_{\mathbb{N}}}|Y \cap [\mathbb{N}]^{\mathbb{N}}) = \aleph_0$. Thus, in some sense, the problem of infinity for the Borel chromatic number of a $\leq\aleph_0$-to-1 Borel function reduces to understanding when the Borel chromatic number of an induced subgraph $\boldsymbol{G}_{S^{\infty}_{\mathbb{N}}}|A$, $A \subseteq [\mathbb{N}]^{\mathbb{N}}$ Borel, is infinite. We thus have the following question, raised in an earlier version of this survey:

Problem 5.22. *For Borel $A \subseteq [\mathbb{N}]^{\mathbb{N}}$ characterize when the induced subgraph $\boldsymbol{G}_{S^{\infty}_{\mathbb{N}}}|A$ has infinite Borel chromatic number.*

In [KST, 8.3] the question was raised whether $\chi_B(\boldsymbol{G}_{S^{\infty}_{\mathbb{N}}}|A)$ is infinite iff $\exists H \in [\mathbb{N}]^{\mathbb{N}}([H]^{\mathbb{N}} \subseteq A)$. This however turns out to be false (see [DPT1]);

letting $\langle\cdot,\cdot\rangle : \mathbb{N}^2 \to \mathbb{N}$ be a bijection, the function given by $f((n_0, n_1, \ldots)) = (\langle n_0, n_1\rangle, \langle n_1, n_2\rangle, \ldots)$ is a Borel homomorphism from $\boldsymbol{G}_{S^\infty_\mathbb{N}}$ to itself, so

$$\chi_B(\boldsymbol{G}_{S^\infty_\mathbb{N}} | \operatorname{ran}(f)) = \aleph_0.$$

However, $\operatorname{ran}(f)$ does not contain any set of the form $[H]^\mathbb{N}$, since removing any number other than $\langle n_0, n_1\rangle$ from the sequence $(\langle n_0, n_1\rangle, \langle n_1, n_2\rangle, \ldots)$ yields a sequence not in $\operatorname{ran}(f)$. Further results concerning Borel chromatic numbers of graphs $\boldsymbol{G}_{S^\infty_\mathbb{N}}|A$ as in Problem 5.22 can be found in [DPT3].

Finally in [TV] the following definitive result was proved:

Theorem 5.23 ([TV, Theorem 1.3])**.** *The set*

$$\{C \subseteq [\mathbb{N}]^\mathbb{N} \colon C \textit{ is closed and } \chi_B(\boldsymbol{G}_{S^\infty_\mathbb{N}}|C) = \aleph_0\}$$

is $\boldsymbol{\Pi}^1_2$-complete (in the space of closed subsets of $[\mathbb{N}]^\mathbb{N}$ with the Effros Borel structure).

This seems to rule out any reasonably explicit positive answer to Problem 5.22.

5.6 Basis problems for graphs with infinite Borel chromatic numbers

Motivated by the $\boldsymbol{G}_0$-dichotomy, the question was raised in [KST, §8] of whether an analogous dichotomy might be true for Borel (or analytic) graphs with infinite Borel chromatic number. To formulate a precise question, let us call (motivated by Conley-Miller [CM1]) a graph $\boldsymbol{G} = (X, G)$ on a standard Borel space *finite color decomposable* if there is a Borel decomposition $X = \bigsqcup_n X_n$ into $E_{\boldsymbol{G}}$-invariant sets with $\chi_B(\boldsymbol{G}|X_n) < \infty, \forall n \in \mathbb{N}$. Note that if $\boldsymbol{G}$ is finite color decomposable and $\boldsymbol{H} \preceq_B \boldsymbol{G}$, so is $\boldsymbol{H}$. Thus, since, by Example 5.3, there are clearly Borel graphs with infinite Borel chromatic number which are finite color decomposable (e.g., $\bigsqcup_n \boldsymbol{K}_n$, with $\boldsymbol{K}_n$ the clique on n vertices), if $\boldsymbol{G}$ is a Borel graph with $\chi_B(\boldsymbol{G})$ infinite such that $\boldsymbol{G} \preceq_B \boldsymbol{H}$ for any Borel graph $\boldsymbol{H}$ with $\chi_B(\boldsymbol{H})$ infinite, then we must have that $\boldsymbol{G} = \bigsqcup_n \boldsymbol{G}_n$ with $\boldsymbol{G}_n$ Borel, $\chi_B(\boldsymbol{G}_n) < \infty$ and $\chi_B(\boldsymbol{G}_n)$ unbounded. Also for such $\boldsymbol{G}$ we have $\boldsymbol{G} \preceq_B \boldsymbol{H}$ iff $\forall n(\boldsymbol{G}_n \preceq_B \boldsymbol{H})$. So the question of whether there is a minimum under $\preceq_B$ Borel graph of infinite Borel chromatic number translates to the following problem, raised in an earlier version of this survey:

Problem 5.24. *Is there a sequence $(\boldsymbol{G}_n)$ of Borel graphs with $\chi_B(\boldsymbol{G}_n) < \infty$ and $\chi_B(\boldsymbol{G}_n)$ unbounded such that for every Borel graph $\boldsymbol{H}$ with infinite Borel chromatic number and any n, we have that $\boldsymbol{G}_n \preceq_B H$?*

In [KST, 8.1]the question was raised whether the graph $\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}$ of the shift $S_{\mathbb{N}}^{\infty}\colon [\mathbb{N}]^{\mathbb{N}} \to [\mathbb{N}]^{\mathbb{N}}$ (see Example 5.2) might be the minimum Borel graph of infinite Borel chromatic number. Conley-Miller [CM1] pointed out that this fails, since $\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}$ is not finite color decomposable. In fact more generally, for any Borel functions $F_i\colon X \to X, 1 \leq i \leq n$, where X is a standard Borel space, if $\boldsymbol{G} = \boldsymbol{G}_{F_1,\dots,F_n}$ is finite color decomposable, then $\chi_B(\boldsymbol{G}) < \infty$. Because if $X = \bigsqcup_m X_m$ is a Borel decomposition into $E_{\boldsymbol{G}}$-invariant sets with $\chi_B(\boldsymbol{G}|X_m) < \infty$, then, by Remark 5.10, $\chi_B(\boldsymbol{G}|X_m) \leq 3^n$, so $\chi_B(\boldsymbol{G}) \leq 3^n$.

This leads to the question of whether $\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}$ is minimum among Borel graphs with infinite Borel chromatic number which are finite color indecomposable. However in [CM1] it is shown that there is an acyclic, locally finite Borel graph $\boldsymbol{G}_{\mathbb{F}}$ with $\chi_B(\boldsymbol{G}_{\mathbb{F}}) = \chi_\mu(\boldsymbol{G}_{\mathbb{F}}) = \aleph_0$ (for an appropriate μ), which is finite color indecomposable but $\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}} \not\preceq_B \boldsymbol{G}_{\mathbb{F}}$. Another question is whether there is a minimum, under $\preceq_B$, among the graphs of the form $\boldsymbol{G}_F$, F a Borel function, that have infinite Borel chromatic number. Could $\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}$ be such a minimum?

It turned out that Problem 5.24 and the questions in the preceding paragraph have negative answers. First it is shown in [Pe] that there is a closed set $C \subseteq [\mathbb{N}]^{\mathbb{N}}$ such that $\chi_B(\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}|C) = \aleph_0$ but $\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}} \not\preceq \boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}|C$. Finally [TV, Theorem 1.3] shows that there is no sequence $(\mathcal{A}_n)$ of analytically parametrizable families of graphs such that for closed $C \subseteq [\mathbb{N}]^{\mathbb{N}}$ the following holds: $\chi_B(\boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}|C) = \aleph_0$ iff for some sequence (n_i) and $\boldsymbol{G}_{n_i} \in \mathcal{A}_{n_i}$ we have $\boldsymbol{G}_{n_i} \preceq_B \boldsymbol{G}_{S_{\mathbb{N}}^{\infty}}|C$. In particular there is no countable basis for $\preceq_B$ among graphs of the form $\boldsymbol{G}_F$, F a Borel function, that have infinite Borel chromatic number and also Problem 5.24 has a negative answer.

6. Finite Borel chromatic numbers

6.1 The dichotomy theorem for Borel chromatic number at most 2

In Section 4, **(H)** we have seen certain results concerning the property $\chi_B(\boldsymbol{G}) \le 2$, including the dichotomy in Theorem 4.31.

The following analog of the $\boldsymbol{G}_0$-dichotomy is proved in [CMSV]. Below for each sequence $b \in \mathbb{N}^{\mathbb{N}}$, we denote by $\boldsymbol{L}_b = (2^{\mathbb{N}} \sqcup \bigsqcup_{n\in\mathbb{N}} X_n, L_b)$, where $X_n = \{(m,n,c) \colon c \in 2^{\mathbb{N}}, 1 \le m \le b_n\}$, the directed graph whose edges are: $0cL_b1c, 0^n10cL_b(1,n,c), (m,n,c)L_b(m+1,n,c), (b_n,n,c)L_b0^n11c, 1 \le m < b_n, c \in 2^{\mathbb{N}}, n \in \mathbb{N}$.

Theorem 6.1 ([CMSV])**.** *For any analytic graph $\boldsymbol{G} = (X, G)$ on a Polish space X, and $b \in (2\mathbb{N})^{\mathbb{N}}$ with $b_n \to \infty$, exactly one of the following holds:*

(i) $\chi_B(\boldsymbol{G}) \le 2$,

(ii) $\boldsymbol{L}_b \preceq_c \boldsymbol{G}$.

On the other hand in [CMSV] it is also shown that the directed version of the $\boldsymbol{G}_0$-dichotomy, see Section 4, **(G)**, fails in this context. More precisely they prove the following:

Theorem 6.2 ([CMSV])**.** *Let $\boldsymbol{D}$ be an analytic graph on a Polish space with $\chi_B(\boldsymbol{D}) > 2$. Then there is a family $(\boldsymbol{M}_c)_{c\in 2^{\mathbb{N}}}$ of Borel directed graphs on Polish spaces with $\chi_B(\boldsymbol{M}_c) > 2, \boldsymbol{M}_c \preceq_c \boldsymbol{D}$ such that if $\boldsymbol{D}_0$ is a directed Borel graph on a Polish space such that $\boldsymbol{D}_0 \preceq_B \boldsymbol{M}_{c_1}, \boldsymbol{M}_{c_2}$, for some $c_1 \ne c_2$, then $\chi_B(\boldsymbol{D}_0) \le 2$.*

We also note here that there cannot be any reasonable dichotomy result concerning the property $\chi_B(\boldsymbol{G}) \le n$, for any $n > 2$, or for the property $\chi_B(\boldsymbol{G}) < \aleph_0$, in view of Theorem 5.23.

6.2 Bounded degree graphs - Brooks' Theorem

Recall from Proposition 5.4 that if $\boldsymbol{G}$ is an analytic graph with $\Delta(\boldsymbol{G}) \leq d$, then $\chi_B(\boldsymbol{G}) \leq d+1$. It is of course trivial that for every $n \in \{1, \dots, d+1\}$ there is a finite graph $\boldsymbol{G}$ with $\Delta(\boldsymbol{G}) \leq d$, whose chromatic number is equal to n. However the problem of finding *acyclic* Borel graphs of bounded degree $\leq d$, whose Borel chromatic number is any given number $n \in \{1, \dots, d+1\}$ is more difficult. Answering a question in [KST, 3.3], Laczkovich (see [KST, Appendix]) found the first examples of acyclic Borel graphs $\boldsymbol{G}$ with $\chi_B(\boldsymbol{G}) = n$ for any prescribed finite value $n \in \{1, 2, \dots\}$. His proof used arguments from dynamics. However these graphs were not even locally finite. The next step was taken in Conley-Kechris [CK, 2.5] who, answering a question of Miller [M08], showed that there are bounded degree acyclic Borel graphs $\boldsymbol{G}$ with $\chi_B(\boldsymbol{G}) = n$, for any prescribed finite value $n \in \{1, 2, \dots\}$. These graphs had $\Delta(\boldsymbol{G})$ of the order of n^2. The proof in [CK] used ergodic theoretic arguments. Finally Marks [Ma1] found a complete answer to this problem, using game theoretic arguments and Borel determinacy.

Theorem 6.3 ([Ma1, 3.2])**.** *Let $d \geq 1$ and $n \in \{1, \dots, d+1\}$. There is a d-regular acyclic Borel graph $\boldsymbol{G}$ with $\chi_B(\boldsymbol{G}) = n$.*

A classical theorem of Brooks (see, e.g., [Di, 5.2.4]) shows that for a finite graph $\boldsymbol{G}$ with $\Delta(\boldsymbol{G}) \leq d$ one actually has $\chi(\boldsymbol{G}) \leq d$, except for two trivial exceptions.

Theorem 6.4 (Brooks)**.** *Let $\boldsymbol{G}$ be a finite graph with $\Delta(\boldsymbol{G}) \leq d$. Then $\chi(\boldsymbol{G}) \leq d$ unless $\boldsymbol{G}$ contains a $(d+1)$-clique, i.e., the complete graph $\boldsymbol{K}_{d+1}$, or $d = 2$ and $\boldsymbol{G}$ contains an odd cycle (i.e., is not bipartite).*

Theorem 6.3 shows that Brooks' Theorem does not hold in general in the Borel context. However it does hold under additional assumptions on the graph $\boldsymbol{G}$.

Very recently Conley, Marks and Tucker-Drob [CMT-D] found a criterion that guarantees the validity of the Brooks bound. A subset S of a graph $\boldsymbol{G}$ is said to be *biconnected* if $\boldsymbol{G}|S$ remains connected after removing any single vertex from S (we do not regard a single vertex as biconnected). A *Gallai tree* is a connected graph whose maximal biconnected sets are all complete graphs or odd cycles. This notion was introduced in the context of studying list colorings, which we discuss in Section 7. Finite Gallai trees are precisely the graphs that are not *degree-colorable*; see [CR, Section 8].

Theorem 6.5 ([CMT-D, Theorem 4.1])**.** *Let $\boldsymbol{G}$ be a Borel graph on a standard Borel space with $\Delta(\boldsymbol{G}) \leq d$, such that every connected component of $\boldsymbol{G}$ is not a Gallai tree. Then $\chi_B(\boldsymbol{G}) \leq d$.*

Recall that a connected locally finite graph $\boldsymbol{G} = (X, G)$ has *at most n ends* if for any finite $F \subseteq X$, the graph $\boldsymbol{G}|(X \setminus F)$ has at most n infinite connected components. Also $\boldsymbol{G}$ has (exactly) *n ends* if n is the least such that $\boldsymbol{G}$ has at most n ends. The graph $\boldsymbol{G}$ has infinitely many ends if for any n it does not have at most n ends.

Theorem 6.5 also implies the following result, proved earlier in [CK].

Theorem 6.6 ([CK, 5.11])**.** *Let $\boldsymbol{G}$ be a Borel graph on a standard Borel space with $\Delta(\boldsymbol{G}) \leq d$, which is vertex transitive and whose connected components have one end. Then $\chi_B(\boldsymbol{G}) \leq d$.*

Also using Theorem 6.5 the authors prove the following full analog of Brooks' Theorem in the measurable and Baire category case.

Theorem 6.7 ([CMT-D, Theorem 1.2])**.** *Let $\boldsymbol{G}$ be a Borel graph on a Polish space with $\Delta(\boldsymbol{G}) \leq d$, where $d \geq 3$. Then, unless $\boldsymbol{G}$ contains a $(d+1)$-clique, we have that $\chi_M(\boldsymbol{G}) \leq d$, and also $\chi_{BM}(\boldsymbol{G}) \leq d$.*

Moreover in [CMT-D, Theorem 6.1] a complete characterization is also given for when a Borel graph $\boldsymbol{G}$ with bounded degree ≤ 2, that does not contain an odd cycle, has χ_M or $\chi_{BM} \leq 2$, thereby extending Brooks' Theorem in this case as well.

Remark 6.8. Earlier the weaker version of Theorem 6.7 with $\chi_\mu^{ap}(\boldsymbol{G}) \leq d$ instead of $\chi_\mu(\boldsymbol{G}) \leq d$ has been established in [CK, 2.19]. It was also shown in [CK, 2.20] that the full version of Brooks' Theorem holds for χ_μ^{ap} in the $d = 2$ case: If $\boldsymbol{G}$ is a Borel graph on a standard Borel space with $\Delta(\boldsymbol{G}) \leq 2$ and $\boldsymbol{G}$ is bipartite, then $\chi_M^{ap}(\boldsymbol{G}) \leq 2$.

The Lovász local lemma is an important tool in probabilistic combinatorics and has many applications to coloring problems. Bernshteyn has proven a measurable version of the Lovász local lemma, and applied it to a number of problems in measurable graph coloring. If $\boldsymbol{G}$ is a graph, we denote by $g(\boldsymbol{G})$ the *girth of $\boldsymbol{G}$*, i.e. the minimum size of a cycle in $\boldsymbol{G}$. If

$\boldsymbol{G}$ is acyclic, then we define $g(\boldsymbol{G}) = \infty$. For graphs of sufficiently large girth, we have the following upper bound on the approximate measure chromatic number.

Theorem 6.9 ([Ber, Theorem 1.4]). *Let $\boldsymbol{G}$ be a Borel graph with $\Delta(\boldsymbol{G}) = d$. If $g(\boldsymbol{G}) \geq 4$, then*

$$\chi_M^{ap}(G) = O\left(\frac{d}{\log d}\right)$$

and if $g(\boldsymbol{G}) \geq 5$, then

$$\chi_M^{ap}(G) = (1 + o(1))\frac{d}{\log d}.$$

6.3 Bounded degree graphs - Vizing's Theorem

We next discuss edge chromatic numbers. In Proposition 5.4 we have seen that if $\boldsymbol{G}$ is an analytic graph with $\Delta(\boldsymbol{G}) \leq d$, where $d \geq 1$, then $\chi'_B(\boldsymbol{G}) \leq 2d - 1$. On the other hand, we have the following classical theorem of Vizing (see, e.g., [Di, 5.3.2]).

Theorem 6.10 (Vizing). *If $\boldsymbol{G}$ is a graph with $\Delta(\boldsymbol{G}) \leq d$, then $\chi'(\boldsymbol{G}) \leq d + 1$.*

Remark 6.11. A theorem of König also shows that for any bipartite graph $\boldsymbol{G}$, we have $\chi'(\boldsymbol{G}) = \Delta(\boldsymbol{G})$ (see [Di, 5.3.1]).

In [KST, p. 15] the question was raised of whether the Vizing bound works for Borel chromatic numbers. This was finally resolved in Marks [Ma1], using again game theoretic methods.

Theorem 6.12 ([Ma1]). *For every $d \geq 1$ and $n \in \{d, \dots, 2d-1\}$, there is a d-regular acyclic Borel graph on a standard Borel space which has $\chi_B(\boldsymbol{G}) = 2$ and $\chi'_B(\boldsymbol{G}) = n$.*

It is open whether Vizing's Theorem holds for measure or Baire measurable chromatic numbers.

Problem 6.13. *Let $\boldsymbol{G} = (X, G)$ be a Borel graph on a Polish space with $\Delta(\boldsymbol{G}) \leq d$. Is it true that $\chi'_M(\boldsymbol{G}) \leq d + 1$? Is it true that $\chi'_{BM}(\boldsymbol{G}) \leq d + 1$?*

The following partial results are known concerning this problem. For a locally countable Borel graph $\boldsymbol{G} = (X, G)$ on a standard Borel space X and a Borel probability measure μ on X, we say that $\boldsymbol{G}$ is *μ-measure preserving* if for some (equivalently any) Borel involutions (T_n) with $\boldsymbol{G} = \boldsymbol{G}_{(T_n)}$, each T_n is μ-measure preserving (see here Section 3,**(A)**). Csóka, Lippner and Pikhurko very recently proved:

Theorem 6.14 ([CLP])**.** *Let $\boldsymbol{G} = (X, G)$ be a Borel graph on a standard Borel space X with $\Delta(\boldsymbol{G}) \leq d$ and let μ be a Borel probability measure on X. If $\boldsymbol{G}$ is μ-measure preserving, then:*
(i) $\chi'_\mu(\boldsymbol{G}) \leq d + \mathrm{O}(\sqrt{d})$,
(ii) If $\boldsymbol{G}$ is bipartite, then $\chi'_\mu(\boldsymbol{G}) \leq d + 1$.

Moreover it is shown in [CLP] that if a certain conjecture on finite graphs is true, then in (i) of 6.14 one obtains the optimal $\chi'_\mu(\boldsymbol{G}) \leq d + 1$. Earlier Marks had proved the following:

Theorem 6.15 ([Ma1, 1.8])**.** *Let $\boldsymbol{G}$ be a Borel graph on a Polish space with $\chi_B(\boldsymbol{G}) \leq 2$ and $\Delta(\boldsymbol{G}) \leq 3$. Then $\chi'_M(\boldsymbol{G}) \leq 4$ and $\chi'_{BM}(\boldsymbol{G}) \leq 4$.*

Bernshteyn has shown that the following holds in the more general case when $\boldsymbol{G}$ is not necessarily μ-measure preserving, but for the approximate edge measure chromatic numbers (defined in [Ber, page 6]):

Theorem 6.16 ([Ber, Theorem 1.5])**.** *Let $\boldsymbol{G}$ be a Borel graph with $\Delta(\boldsymbol{G}) = d$. Then*

$$\chi_M'^{,ap}(\boldsymbol{G}) = d + o(d).$$

Bernshteyn's proof uses a measurable verison of the Lovász local lemma. This technique can also be used to give a new proof of [CLP] in the case where $\boldsymbol{G}$ is the graph arising from the shift action of a group Γ on the space $[0,1]^\Gamma$. Moreover, in this case Bernshteyn proves a list coloring version of this theorem, in the case when the list of colors depends only on the group element associated to the edge (see [Ber, Theorem 1.3]).

Addendum. It has been now shown in [GrP] that if $\boldsymbol{G} = (X, G)$ is a Borel graph on a standard Borel space X with $\Delta(\boldsymbol{G}) \leq d$ and μ is a Borel probability measure on X such that $\boldsymbol{G}$ is μ-measure preserving, then $\chi'_\mu(\boldsymbol{G}) \leq d + 1$, It is also shown in that paper that $\chi_M'^{,ap}(\boldsymbol{G}) \leq d$. Moreover these results are extended to Borel multi-graphs (in which multiple edges between adjacent vertices are allowed).

In the Baire measurable setting, we have an analog of König's bound (see Remark 6.11) for acyclic, regular graphs.

Theorem 6.17. *Suppose $\boldsymbol{G}$ is an acyclic, d-regular Borel graph in a Polish space, where $d \geq 3$. Then $\chi'_{BM}(\boldsymbol{G}) = d$.*

For a sketch of a proof, see the last paragraph of Section 15.

Concerning shift directed graphs $\boldsymbol{sD}$ as in Section 3,**(A)**, a result in finite graph theory asserts that if $\boldsymbol{D}$ is a finite directed graph with $\chi(\boldsymbol{D}) = n$, then $\log_2 n \leq \chi(\boldsymbol{sD}) \leq \min\{k \colon n \leq \binom{k}{\llcorner k/2 \lrcorner}\}$ (see [HE], [N1]).

It is not hard to verify that the proofs in the finite case given in [HE, Section 4] or [N1, 5.6] also show that for any locally countable directed Borel graph $\boldsymbol{D}$ with $\chi_B(\boldsymbol{D}) = n$, we also have $\log_2 n \leq \chi_B(\boldsymbol{sD}) \leq \min\{k \colon n \leq \binom{k}{\llcorner k/2 \lrcorner}\}$.

6.4 Hyperfinite graphs

We call a locally countable Borel graph on a standard Borel space *hyperfinite* (resp., *μ-hyperfinite*) if the equivalence relation $E_{\boldsymbol{G}}$ is hyperfinite (resp., μ-hyperfinite). For such graphs we have the following result:

Theorem 6.18 ([CM2])**.** *Let $\boldsymbol{G} = (X, G)$ be a locally finite Borel graph on a standard Borel space X, with $\chi(\boldsymbol{G}) < \infty$, and μ a probability measure on X, for which $\boldsymbol{G}$ is μ-hyperfinite. Then*

$$\chi_\mu(\boldsymbol{G}) \leq 2\chi(\boldsymbol{G}) - 1.$$

In particular, if $\boldsymbol{G}$ is bipartite (e.g., acyclic), then $\chi_\mu(\boldsymbol{G}) \leq 3$, and if $F_1, \dots, F_n$ are $< \aleph_0$-to-1 Borel functions and $\boldsymbol{G} = \boldsymbol{G}_{F_1,\dots,F_n}$, then $\chi_\mu(\boldsymbol{G}) \leq 4n + 1$.

In view of Theorem 5.6 and Theorem 6.18, we have the following open problem of Conley and Miller:

Problem 6.19. *Let $\boldsymbol{G} = (X, G)$ be a bounded degree Borel graph on a standard Borel space X, with $\chi(\boldsymbol{G}) < \infty$, which is hyperfinite. Is it true that $\chi_B(\boldsymbol{G}) \leq 2\chi(\boldsymbol{G}) - 1$?*

For the acyclic case, Theorem 6.18 holds even when $\boldsymbol{G}$ is locally countable, see [M08, 3.1]. We will see in Theorem 6.59 below that the hypothesis of hyperfiniteness cannot be removed from this theorem, even when $\boldsymbol{G}$ is acyclic of bounded degree.

On the other hand the following holds:

Theorem 6.20 ([CK, 3.1])**.** *Let $\boldsymbol{G}$ be a locally countable, acyclic Borel graph on a standard Borel space X and μ a Borel probability measure on X for which $\boldsymbol{G}$ is μ-hyperfinite. Then $\chi_\mu^{ap}(\boldsymbol{G}) \leq 2$.*

However this fails if $\boldsymbol{G}$ is only bipartite instead of acyclic. The graph $\boldsymbol{G}_0'$ introduced in Section 3,**(C)** has $i_\mu(\boldsymbol{G}_0') = 0$, thus $\chi_\mu^{ap}(\boldsymbol{G}_0') = 2^{\aleph_0}$. It holds though in the bipartite case if moreover $\boldsymbol{G}$ is locally finite. In fact, more generally we have the following result:

Proposition 6.21 ([CK, 3.8])**.** *Let G be a locally finite Borel graph on a standard Borel space X and μ a Borel probability measure on X for which $\boldsymbol{G}$ is μ-hyperfinite. Then $\chi_\mu^{ap}(\boldsymbol{G}) \leq$ the minimum of all $\chi(\boldsymbol{G}|A)$, with A an $E_{\boldsymbol{G}}$-invariant Borel set of μ-measure 1, and if $\boldsymbol{G}$ is μ-measure preserving we have equality.*

Finally one can ask if Brooks' Theorem holds for bounded degree Borel graphs which are hyperfinite. Clearly the acyclic graph $\boldsymbol{G}$ generated by the free part of the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$ satisfies $\Delta(\boldsymbol{G}) = 2$ but $\chi_B(\boldsymbol{G}) = \chi_\mu(\boldsymbol{G}) = \chi_{BM}(\boldsymbol{G}) = 3$ (for μ the usual product measure), so Brooks' Theorem fails in this case. However the following problem is open:

Problem 6.22. *Let $d \geq 3$. Is it true that if $\boldsymbol{G}$ is a d-regular, acyclic, hyperfinite Borel graph, then $\chi_B(\boldsymbol{G}) \leq d$?*

The following is also open:

Problem 6.23. *Let $d \geq 3$. Is there a d-regular, hyperfinite Borel graph $\boldsymbol{G}$ such that $\chi_B(\boldsymbol{G}) > \chi(\boldsymbol{G}) + 1$?*

6.5 Graphs generated by group actions

An important class of graphs of bounded degree consists of the graphs induced by actions of finitely generated groups.

Let Γ be a finitely generated group with a finite symmetric set of generators S, with $1 \notin S$. We will call (Γ, S) a *marked group*. Let $a\colon \Gamma \times X \to X$ be a *free* (i.e., $a(\gamma, x) \neq x$, if $\gamma \neq 1$) Borel action of Γ on a standard Borel space X. We usually write $a(\gamma, x) = \gamma \cdot x$, when a is understood. Let $\boldsymbol{G}(S, a)$ be the "Cayley graph" associated with this action: $\boldsymbol{G}(S, a) = (X, G(S, a))$, where $xG(S, a)y \Leftrightarrow \exists \gamma \in S(\gamma \cdot x = y)$. Clearly this is a Borel graph which is d-regular, where $d = |S|$. Each connected component of $\boldsymbol{G}(S, a)$ is isomorphic to the (right) Cayley graph of (Γ, S) denoted by $\mathbf{Cay}(\Gamma, S) = (\Gamma, \mathrm{Cay}(\Gamma, S))$, where $\delta \mathrm{Cay}(\Gamma, S)\epsilon \Leftrightarrow \exists \gamma \in S(\delta\gamma = \epsilon)$. The isomorphism is given by the map $\gamma \mapsto \gamma^{-1} \cdot x$, where x is in the connected component.

Of particular interest are the shift actions of countable groups. Let X be a standard Borel space with $|X| \geq 2$. The *shift action* of a countable group Γ on the space X^Γ, denoted by $s_{\Gamma,X}$, is given by $\gamma \cdot p(\delta) = p(\gamma^{-1}\delta)$, for $\gamma, \delta \in \Gamma, p \in X^\Gamma$. This action is not free, so we let $F(X^\Gamma) = \{p \in X^\Gamma \colon \forall \gamma \neq 1(\gamma \cdot p \neq p)\}$ be the *free part* of the action, which is a Γ-invariant Borel set. We denote by $\boldsymbol{G}(S, X^\Gamma) = \boldsymbol{G}(S, s_{\Gamma,X})$ the corresponding graph on the free part.

We note that for any free Borel action $a \colon \Gamma \times X \to X$, there is a Γ-*equivariant* Borel embedding of X into $F([0,1]^\Gamma)$ (in fact even to $F(\mathbb{N}^\Gamma)$ – see [JKL, 4.2, 5.4]), i.e., a Borel injection $\pi \colon X \to F([0,1]^\Gamma)$ such that $\pi(\gamma \cdot x) = \gamma \cdot \pi(x)$. In particular the graph $\boldsymbol{G}(S, a)$ is Borel isomorphic to the induced subgraph of $\boldsymbol{G}(S, [0,1]^\Gamma)$ on a $E_{\boldsymbol{G}(S,[0,1]^\Gamma)}$-invariant Borel set. We thus put

$$\boldsymbol{G}_\infty(\Gamma, S) = \boldsymbol{G}(S, [0,1]^\Gamma).$$

Therefore

$$\chi_B(\boldsymbol{G}(S, a)) \leq \chi_B(\boldsymbol{G}_\infty(\Gamma, S)),\ \chi'_B(\boldsymbol{G}(S, a)) \leq \chi'_B(\boldsymbol{G}_\infty(\Gamma, S)),$$

for any free action a.

It is not possible in general to Borel embed a free Borel action, e.g., of $\Gamma = \mathbb{Z}$, into $F(k^\Gamma)$ for finite k. Very recently Seward and Tucker-Drob [ST] proved that for any infinite Γ and any free Borel action $a \colon \Gamma \times X \to X$, there is a Γ-equivariant Borel map $\pi \colon X \to F(2^\Gamma)$, which is therefore a Borel homomorphism from $\boldsymbol{G}(S, a)$ to $\boldsymbol{G}(S, 2^\Gamma)$, for any S. Thus we have the following result, which answers a question of Marks [Ma1]:

Theorem 6.24 ([ST]). *For any infinite marked group (Γ, S) and any standard Borel space X, with $|X| \geq 2$, we have*

$$\chi_B(\boldsymbol{G}(S, X^\Gamma)) = \chi_B(\boldsymbol{G}_\infty(\Gamma, S)) \text{ and } \chi'_B(\boldsymbol{G}(S, X^\Gamma)) = \chi'_B(\boldsymbol{G}_\infty(\Gamma, S)).$$

Theorem 6.5 implies the following version of Brooks' Theorem for graphs generated by group actions:

Theorem 6.25 ([CMT-D]). *For any infinite marked group (Γ, S), with $|S| = d$, such that* **Cay**(Γ, S) *is not a Gallai tree and any free Borel action a of Γ on a standard Borel space X, we have*

$$\chi_B(\boldsymbol{G}(S, a)) \leq d.$$

It is an interesting open question whether the converse of Theorem 6.25 is true. If correct this would give a complete characterization for the validity of the Brooks' bound for graphs generated by group actions.

Problem 6.26 ([CMT-D]). *For any infinite marked group (Γ, S), with $|S| = d$, is it the case that* **Cay**(Γ, S) *is not a Gallai tree iff for any free Borel action a of Γ on a standard Borel space X we have*

$$\chi_B(\boldsymbol{G}(S, a)) \leq d.$$

Some special cases of this problem are settled by Theorem 6.31. In particular, the problem has a positive answer for groups that are finite free products of $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$, where $\mathbb{Z}$ is generated as usual and each $\mathbb{Z}/n\mathbb{Z}$ is generated by all its nontrivial group elements. Of course, the Cayley graphs of such groups are Gallai trees.

Recall that in an infinite, finitely generated group Γ, we define the *number of ends* of Γ to be equal to the number of ends of **Cay**(Γ, S), for any finite symmetric generating set S (this is independent of S; see [Me, Theorem 11.23]). The number of ends is 1, 2 or ∞ (see [D, IV.25, (vi)]). The following result, proved originally in [CK] by using Theorem 6.6, also follows from Theorem 6.25:

Theorem 6.27 ([CK, 5.12]). *Let (Γ, S) be an infinite marked group with Γ with finitely many ends, which is not isomorphic to $\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. If $|S| = d$, then $\chi_B(\boldsymbol{G}(S, a)) \leq d$ for any free Borel action a of Γ.*

For $\Gamma = \mathbb{Z}$, with $S = \{\pm 1\}$, or $\Gamma = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b | a^2 = b^2 = 1\rangle$, with $S = \{a, b\}$, the graph $\boldsymbol{G}_\infty(\Gamma, S)$ has Borel chromatic number equal to 3. Thus, apart from these two exceptions, the Brooks bound applies in this case. Sometimes this upper bound can be improved. Answering a question in [KST, 4.8], it was proved by Gao and Jackson [GJ, 4.2] (and

Miller for $n = 2$) that for $\mathbb{Z}^n$ and S the usual set of generators for $\mathbb{Z}^n$, $\chi_B(\boldsymbol{G}_\infty(\mathbb{Z}^n, S)) \leq 4$. Very recently the exact Borel chromatic number of $\boldsymbol{G}_\infty(\mathbb{Z}^n, S)$ was computed:

Theorem 6.28 (Gao-Jackson-Krohne-Seward)**.** *For any $n \geq 1$, and S the usual set of generators for $\mathbb{Z}^n$, $\chi_B(\boldsymbol{G}_\infty(\mathbb{Z}^n, S)) = 3$ and thus for any free Borel action a of $\mathbb{Z}^n$ we have that $\chi_B(\boldsymbol{G}(S, a)) \leq 3$.*

Gao-Jackson-Krohne-Seward also showed that the *continuous* chromatic number of the graph $\boldsymbol{G}(S, 2^{\mathbb{Z}^n})$ is equal to 4. Finally they proved that for any countable graph $\boldsymbol{H}$, $\boldsymbol{G}(S, 2^{\mathbb{Z}^n}) \preceq_B \boldsymbol{H}$ iff $\boldsymbol{H}$ is not bipartite.

Remark 6.29. Note that it also follows from Theorem 6.18 that for any free Borel action a of $\mathbb{Z}^n$ on a standard Borel space and S the usual set of generators for $\mathbb{Z}^n$, we have that $\chi_M(\boldsymbol{G}(S, a)) \leq 3$, a result proved earlier by Gao-Jackson-Miller and independently Timár [Ti].

Remark 6.30. It should also be pointed out that when a free Borel action a of Γ admits an invariant Borel probability measure μ with respect to which it is weakly mixing, then $\chi_B(\boldsymbol{G}(S, a)) \geq \chi_\mu(\boldsymbol{G}(S, a)) \geq 3$ (see [CK, page 148] or the argument in Example 4.15).

Not much else seems to be known concerning $\chi_B(\boldsymbol{G}_\infty(\Gamma, S))$, for Γ with finitely many ends.

We note here that by Stallings' Theorem (see [St]) a torsion-free finitely generated group has infinitely many ends iff Γ is a non-trivial free product. In particular, the question was raised in [CK, 5.17] of whether the Brooks bound holds for free Borel actions of the free group $\mathbb{F}_n$ (with a symmetrized free set of generators). This was answered in the negative by Marks [Ma1], using the following result:

Theorem 6.31 ([Ma1, 1.2])**.** *Let $(\Gamma, S), (\Delta, T)$ be infinite marked groups. Then*

$$\chi_B(\boldsymbol{G}_\infty(\Gamma * \Delta, S \sqcup T)) \geq \chi_B(\boldsymbol{G}_\infty(\Gamma, S)) + \chi_B(\boldsymbol{G}_\infty(\Delta, T)) - 1.$$

Corollary 6.32 ([Ma1])**.** *Let $\mathbb{F}_n$ be the free group with n generators and $S = \{\gamma_1^{\pm 1}, \dots, \gamma_n^{\pm 1}\}$, where $\{\gamma_1, \dots, \gamma_n\}$ is a set of free generators. Then $\chi_B(\boldsymbol{G}_\infty(\mathbb{F}_n, S)) = 2n + 1$.*

This also partially confirms Conjecture 6.26. Moreover the following is derived in [Ma1]:

Theorem 6.33 ([Ma1]). *Let $\mathbb{F}_n$ be the free group with n generators and $S = \{\gamma_1^{\pm 1}, \dots, \gamma_n^{\pm 1}\}$, where $\{\gamma_1, \dots, \gamma_n\}$ is a set of free generators. Then for each $2 \leq m \leq 2n+1$, there is a free Borel action a of $\mathbb{F}_n$ with $\chi_B(\boldsymbol{G}(S,a)) = m$.*

The proof of Theorem 6.31 as well as Theorems 6.3 and 6.12 is based on the following Main Lemma in [Ma1], which is proved by game theoretic methods and Borel determinacy.

Theorem 6.34 ([Ma1, Main Lemma 2.1]). *Let Γ, Δ be countable groups. If $A \subseteq F([0,1]^{\Gamma * \Delta})$ is a Borel set, then one of the following holds:*

(i) There is a Γ-equivariant continuous injection

$$\pi\colon F([0,1]^{\Gamma}) \to F([0,1]^{\Gamma * \Delta})$$

such that $\pi(F([0,1]^{\Gamma})) \subseteq A$,

(ii) There is a Δ-equivariant continuous injection

$$\rho\colon F([0,1]^{\Delta}) \to F([0,1]^{\Gamma * \Delta})$$

such that $\rho(F([0,1]^{\Delta})) \subseteq F([0,1]^{\Gamma * \Delta}) \setminus A$.

Note here that Γ acts on $F([0,1]^{\Gamma * \Delta})$ by restricting the shift action to the subgroup Γ of $\Gamma * \Delta$ and similarly for Δ.

It is also shown in [Ma1, 3.4] that $\chi_B(\boldsymbol{G}_\infty(\Gamma * \Delta, S \sqcup T)) \leq \chi_B(\boldsymbol{G}_\infty(\Gamma, S)) \cdot \chi_B(\boldsymbol{G}_\infty(\Delta, T))$. The following are open problems:

Problem 6.35 ([Ma1, 3.3, 3.5]). *(i) Are there marked groups $(\Gamma, S), (\Delta, T)$ for which*

$$\chi_B(\boldsymbol{G}_\infty(\Gamma * \Delta, S \cup T)) > \chi_B(\boldsymbol{G}_\infty(\Gamma, S)) + \chi_B(\boldsymbol{G}_\infty(\Delta, T)) - 1?$$

(ii) Are there non-trivial marked groups $(\Gamma, S), (\Delta, T)$ for which

$$\chi_B(\boldsymbol{G}_\infty(\Gamma * \Delta, S \cup T)) = \chi_B(\boldsymbol{G}_\infty(\Gamma, S)) \cdot \chi_B(\boldsymbol{G}_\infty(\Delta, T))?$$

Concerning measure or Baire measurable chromatic numbers, we have the following corollary of Theorem 6.7:

Theorem 6.36 ([CMT-D]). *Let (Γ, S) be an infinite marked group with $|S| = d \geq 3$. Then for any free Borel action a of Γ on a Polish space, we have $\chi_M(\boldsymbol{G}(S,a)) \leq d$ and $\chi_{BM}(\boldsymbol{G}(S,a)) \leq d$.*

Remark 6.37. This clearly fails for $|S| = 2$, since, for example, for $(\Gamma, S) = (\mathbb{Z}, \{\pm 1\})$, and a the free part of the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$, we have

$$\chi_\mu(\boldsymbol{G}(S, a)) = 3,$$

where μ is the usual product measure. Moreover we also have

$$\chi_{BM}(\boldsymbol{G}(S, a)) = 3.$$

However, since the only infinite marked groups (Γ, S) with $|S| = 2$ are $\mathbb{Z}$, with $S = \{\pm 1\}$, and $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b | a^2 = b^2 = 1\rangle$, with $S = \{a, b\}$, it is clear from Remark 6.8 that for *every* infinite marked group (Γ, S) with $|S| = d$, any free Borel action a of Γ on a standard Borel space X and any Borel probability measure μ on X, we have $\chi^{ap}_\mu(\boldsymbol{G}(S, a)) \leq d$.

We next mention the following interesting question:

Problem 6.38. *Let $(\Gamma, S), (\Delta, T)$ be two infinite marked groups with isomorphic Cayley graphs. Is it true that the Borel chromatic numbers of $\boldsymbol{G}_\infty(\Gamma, S)$ and $\boldsymbol{G}_\infty(\Delta, T)$ are equal?*

We point out that this fails if instead of the Borel chromatic number one considers the Borel edge chromatic number. As in Remark 6.37, if we take $(\Gamma, S) = (\mathbb{Z}, \{\pm 1\})$ and (Δ, T) with $\Delta = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) = \langle a, b | a^2 = b^2 = 1\rangle$ and $T = \{a, b\}$, then the Cayley graphs of these two marked groups are isomorphic but $\chi'_B(\boldsymbol{G}_\infty(\mathbb{Z}, S)) = 3$ and $\chi'_B(\boldsymbol{G}_\infty(\Delta, T)) = 2$; see [KST, pp. 19–20]. One can also raise the same question as in Problem 6.38 for measurable or approximately measurable chromatic numbers.

Addendum. Problem 6.38 has now been solved by Weilacher [Wei], who constructed pairs $(\Gamma, S), (\Delta, T)$ of infinite marked groups with isomorphic Cayley graphs but for which the Borel chromatic numbers of $\boldsymbol{G}_\infty(\Gamma, S)$ and $\boldsymbol{G}_\infty(\Delta, T)$ are different. Similarly for measurable chromatic numbers.

Finally, in general, not much seems to be known concerning Borel edge chromatic numbers of the graphs $\boldsymbol{G}(S, a)$, except the obvious lower bound d, where $|S| = d$. For example, we have the following open problem.

Problem 6.39. *Let $\mathbb{F}_n$ be the free group with $n \geq 2$ generators and let $S = \{\gamma_1^{\pm 1}, \dots, \gamma_n^{\pm 1}\}$, with $\{\gamma_1, \dots, \gamma_n\}$ a set of free generators. What is the Borel edge chromatic number $\chi'_B(\boldsymbol{G}_\infty(\mathbb{F}_n, S))$? Similarly for $\mathbb{Z}^n$.*

For $(\Gamma, S) = (\mathbb{Z}, \{\pm 1\})$, $\chi'_B(\boldsymbol{G}(S,a)) \leq 3$ for any free Borel action a, from which it easily follows that $\chi'_B(\boldsymbol{G}_\infty(\mathbb{F}_n, S)) \leq 3n$, thus

$$2n \leq \chi'_B(\boldsymbol{G}_\infty(\mathbb{F}_n, S)) \leq 3n.$$

Notice also that there are free Borel actions a of $\mathbb{F}_n$ for which we have $\chi'_B(\boldsymbol{G}(S,a)) = 2n$. To see this, take any free action b on X and define the action a on $X \times \mathbb{Z}/2\mathbb{Z}$ by $a(\gamma, (x,i)) = (b(\gamma, x), \varphi(\gamma) + i)$, where φ is the homomorphism from $\mathbb{F}_n$ to $\mathbb{Z}/2\mathbb{Z}$ that sends S to 1. Then color each edge $\{(x,0),(y,1)\}$ of $\boldsymbol{G}(S,a)$ by the unique $s \in S$ such that $b(s,x) = y$.

Also note that, if instead of the free groups $\mathbb{F}_n$, we use the free products $\Delta_d = (\mathbb{Z}/2\mathbb{Z})^{*d} = \langle a_1, \dots, a_d |\ a_1^2 = \cdots = a_d^2 = 1\rangle$ and $T = \{a_1, \dots, a_d\}$, then $\chi'_B(\boldsymbol{G}(T,a)) = d$, for any free Borel action a of Δ_n. In particular, for $d = 2n$, $\chi'_B(\boldsymbol{G}_\infty(\Delta_{2n}, T)) = 2n$, while the Cayley graphs of $(\mathbb{F}_n, S), (\Delta_{2n}, T)$ are isomorphic.

On the other hand, for measurable edge chromatic numbers, Theorem 6.14 implies the following result:

Theorem 6.40 ([CLP]). *Let (Γ, S) be an infinite marked group with $|S| = d$, let a be a free Borel action of Γ on a standard Borel space X and let μ be a Borel probability measure on X which is a-invariant. Then $\chi'_\mu(\boldsymbol{G}(S,a)) \leq d + \mathrm{O}(d)$ and if $\mathbf{Cay}(\Gamma, S)$ is bipartite, then $\chi'_\mu(\boldsymbol{G}(S,a)) \leq d+1$.*

Addendum. It now follows from the results of [GP] that in Theorem 6.40 one always has $\chi'_\mu(\boldsymbol{G}(S,a)) \leq d+1$.

Also Theorem 6.15 implies that if (Γ, S) is an infinite marked group with $|S| = 3$ and a is a free Borel action of Γ on a Polish space X, with $\chi_B(\boldsymbol{G}(S,a)) \leq 2$, then $\chi'_M(\boldsymbol{G}(S,a)) \leq 4$ and $\chi'_{BM}(\boldsymbol{G}(S,a)) \leq 4$.

Taking in Theorem 6.40 the group $\Gamma = \mathbb{F}_n$ with $S = \{\gamma_1^{\pm 1}, \dots, \gamma_n^{\pm 1}\}$, with$\{\gamma_1, \dots, \gamma_n\}$ a set of free generators, and any free Borel action a of $\mathbb{F}_n$ on a standard space X and a Borel probability measure μ on X, which is a-invariant, we have that $2n \leq \chi'_\mu(\boldsymbol{G}(S,a)) \leq 2n+1$ (and by the comments in the paragraph following Problem 6.39 the lower bound is achieved by some measure preserving action). The following is an open problem:

Problem 6.41. *Let $\mathbb{F}_n$ be the free group with $n \geq 2$ generators and let $S = \{\gamma_1^{\pm 1}, \dots, \gamma_n^{\pm 1}\}$, where $\{\gamma_1, \dots, \gamma_n\}$ is a set of free generators. What is the μ-measurable edge chromatic number $\chi'_\mu(\boldsymbol{G}_\infty(\mathbb{F}_n, S))$, where μ is the usual product measure on $[0,1]^{\mathbb{F}_n}$?*

We saw in Theorem 6.17 that $\chi'_{BM}(\boldsymbol{G}_\infty(\mathbb{F}_n, S)) = 2n$.

In the recent paper [Ber1], the author considers, for a given countable group Γ and continuous action a of Γ on a Polish space X, Baire measurable maps $f\colon X \to \mathbb{N}$, viewed as abstract Baire measurable "colorings". Any such map gives rise to the function $F\colon X \to \mathbb{N}^\Gamma$ given by $F(x)(\gamma) = f(\gamma^{-1} \cdot x)$. This is Baire measurable and Γ-equivariant, where Γ acts on $\mathbb{N}^\Gamma$ by shift. Given a *subshift* $S \subseteq \mathbb{N}^\Gamma$, i.e., a non-empty, closed, shift-invariant subset of $\mathbb{N}^\Gamma$, we say that f is an *S-map* if F maps a comeager subset of X into S.

For example, assume that (Γ, S) is an infinite marked group, $k \geq 1$, and let $\mathrm{Col}(\Gamma, S, k)$ be the subshift consisting of all k-colorings of $\mathbf{Cay}(\Gamma, S)$. Then a $\mathrm{Col}(\Gamma, S, k)$-map for a free action a as above is a Baire measurable coloring of $\boldsymbol{G}(S, a)$ on a comeager set.

For each action a as above, we denote by $\mathrm{Sh}_{BM}(a, \mathbb{N})$ the set of all subshifts S for which there is an S-map. The following is proved in [Ber1].

Theorem 6.42 ([Ber1, Theorem 2.3])**.** *Let a be a free continuous action of a countable group Γ on a Polish space X. Then we have:*

(i) If the equivalence relation E_a induced by a is generically smooth (i.e., it is smooth on a comeager set), then $\mathrm{Sh}_{BM}(a, \mathbb{N})$ contains all the subshifts.

(ii) If E_a is not generically smooth, then $\mathrm{Sh}_{BM}(a, \mathbb{N})$ is a complete analytic set (in the space of closed subsets of $\mathbb{N}^\Gamma$ with the Effros Borel structure).

Moreover for the shift action $s = s_{\Gamma,[0,1]}$ the author proves a combinatorial characterization of $\mathrm{Sh}_{BM}(s, \mathbb{N})$. This allows the application of results from finite combinatorics to calculation of bounds of Baire measurable chromatic numbers. For example the following is proved in [Ber1]:

Theorem 6.43 ([Ber1, Corollary 2.11])**.** *Let (Γ, S) be an infinite marked group with $\mathbf{Cay}(\Gamma, S)$ planar. Then $\chi_{BM}(\boldsymbol{G}_\infty(\Gamma, S)) \leq 3$, if $\mathbf{Cay}(\Gamma, S)$ has no cycles of length 3 and 4; $\chi_{BM}(\boldsymbol{G}_\infty(\Gamma, S)) \leq 4$, if $\mathbf{Cay}(\Gamma, S)$ has cycles of length 4 but not of length 3; and $\chi_{BM}(\boldsymbol{G}_\infty(\Gamma, S)) \leq 5$, otherwise.*

6.6 Measure preserving group actions

Throughout **(F)** we let (Γ, S) be an *infinite* marked group, i.e., Γ is an infinite group and S a finite symmetric set of generators with $1 \notin S$. Let (X, μ)

be a *standard measure space*, i.e., X is a standard Borel space and μ a non-atomic Borel probability measure on X. All such spaces are isomorphic, so we can freely move among them. We will consider measure preserving Borel actions a of Γ on (X, μ). We identify two such actions a, a', if for all γ, we have $a(\gamma, x) = a'(\gamma, x), \mu$-a.e.. We will write $\boldsymbol{a}$ for the equivalence class of a under this identification but still refer to $\boldsymbol{a}$ as a measure preserving action and write $\boldsymbol{a}(\gamma, x)$ if there is no danger of confusion, where we implicitly ignore null sets.

Let $A(\Gamma, X, \mu)$ be the space of such actions. Equipped with the *weak topology* this is a Polish space (see, e.g., [K10]). We denote by $\mathrm{FR}(\Gamma, X, \mu)$ the subspace of *free* actions, where $\boldsymbol{a}$ is free iff for all $\gamma \neq 1$, we have $\boldsymbol{a}(\gamma, x) \neq x, \mu$-a.e. This is a G_δ subset of $A(\Gamma, X, \mu)$, so also a Polish space. For every $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, we define the "Cayley graph" $\boldsymbol{G}(S, \boldsymbol{a})$ as in Section 5, **(D)**, noting that it is defined only μ-a.e., that is for any two representatives a, a' of $\boldsymbol{a}$, the graphs $\boldsymbol{G}(S, a)$, $\boldsymbol{G}(S, a')$ are defined and equal on a set of measure 1, which is invariant under both a and a'. In particular this means that the parameters $i_\mu(\boldsymbol{G}(S, \boldsymbol{a})), \chi_\mu(\boldsymbol{G}(S, \boldsymbol{a}))$ and $\chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a}))$ are well-defined and can be viewed as functions on the space $\mathrm{FR}(\Gamma, X, \mu))$, taking values in $[0, 1]$, $\mathbb{N}$ and $\mathbb{N}$, resp.

In the space $\mathrm{FR}(\Gamma, X, \mu))$, we have a hierarchy of complexity of actions induced by the quasi-order of *weak containment*, in symbols $\prec$, defined as follows: Let $\mathrm{Aut}(X, \mu)$ be the Polish group of Borel automorphisms of X that preserve μ (in which we identify two such Borel automorphisms if they agree μ-a.e.). With the weak topology this is a Polish group acting continuously on $\mathrm{FR}(\Gamma, X, \mu)$ as follows. For $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, let $\gamma^{\boldsymbol{a}} \in \mathrm{Aut}(X, \mu)$ be defined by $\gamma^{\boldsymbol{a}}(x) = \boldsymbol{a}(\gamma, x)$. Then for $T \in \mathrm{Aut}(X, \mu), \boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, let $T \cdot \boldsymbol{a} = \boldsymbol{b} \iff T\gamma^{\boldsymbol{a}}T^{-1} = \gamma^{\boldsymbol{b}}, \forall \gamma \in \Gamma$. Then we put

$$\boldsymbol{a} \prec \boldsymbol{b} \iff \boldsymbol{a} \in \overline{\{T \cdot \boldsymbol{b} \colon T \in \mathrm{Aut}(X, \mu)\}},$$

where closure is in the weak topology. We finally say that $\boldsymbol{a}, \boldsymbol{b}$ are *weakly equivalent*, in symbols $\boldsymbol{a} \sim \boldsymbol{b}$, if $\boldsymbol{a} \prec \boldsymbol{b}, \boldsymbol{b} \prec \boldsymbol{a}$. In case $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, $\boldsymbol{b} \in \mathrm{FR}(\Gamma, Y, \nu)$, we also define $\boldsymbol{a} \prec \boldsymbol{b}$ (resp., $\boldsymbol{a} \sim \boldsymbol{b}$) if for some (equivalently any) $\boldsymbol{b}' \in \mathrm{FR}(\Gamma, X, \mu)$ which is isomorphic (measure theoretically) to $\boldsymbol{b}$, we have $\boldsymbol{a} \prec \boldsymbol{b}'$ (resp., $\boldsymbol{a} \sim \boldsymbol{b}'$).

It turns out that $(\mathrm{FR}(\Gamma, X, \mu), \prec)$ has a least element, $\boldsymbol{a}_{\Gamma,0}$, which is the (weak equivalence class of the) shift action of Γ on 2^Γ (Abért-Weiss [AW]). There is also a largest element, $\boldsymbol{a}_{\Gamma,\infty}$ (Glasner-Thouvenot-Weiss [GTW], Hjorth (unpublished)).

We now have:

Theorem 6.44 ([CK, 4.1, 4.2, 4.3])**.** *The map*

$$\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)) \mapsto i_\mu(\boldsymbol{G}(S, \boldsymbol{a}))$$

is lower semicontinuous. In particular, for actions $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ *and* $\boldsymbol{b} \in \mathrm{FR}(\Gamma, Y, \nu)$*, we have*

$$\boldsymbol{a} \prec \boldsymbol{b} \implies i_\mu(\boldsymbol{G}(S, \boldsymbol{a})) \leq i_\nu(\boldsymbol{G}(S, \boldsymbol{b})).$$

Moreover

$$\boldsymbol{a} \prec \boldsymbol{b} \implies \chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a})) \geq \chi_\nu^{ap}(\boldsymbol{G}(S, \boldsymbol{b})).$$

Thus both i_μ *and* χ_μ^{ap} *are invariants of weak equivalence.*

On the other hand χ_μ is not an invariant of weak equivalence. For example, all free actions of $\mathbb{Z}$ are weakly equivalent but there are $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{FR}(\mathbb{Z}, X, \mu)$ with $\chi_\mu(\boldsymbol{G}(S, \boldsymbol{a})) = 2, \chi_\mu(\boldsymbol{G}(S, \boldsymbol{b})) = 3$ (where as usual $S = \{\pm 1\}$).

In Abért-Elek [AE] the authors define a compact metrizable topology on the quotient space $\mathrm{FR}(\Gamma, X, \mu)/\prec$, which is larger than the quotient topology. Peter Burton has shown that the function $[\boldsymbol{a}]_\sim \mapsto i_\mu(\boldsymbol{G}(S, \boldsymbol{a}))$ is continuous in the Abért-Elek topology.

Given actions $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ and $\boldsymbol{b} \in \mathrm{FR}(\Gamma, Y, \nu)$, we say that $\boldsymbol{a}$ is a *factor* of $\boldsymbol{b}$, in symbols $\boldsymbol{a} \sqsubseteq \boldsymbol{b}$, if there is a Borel map $\pi\colon Y \to X$ such that $\pi_*\nu = \mu$ and $\pi(\boldsymbol{b}(\gamma, y)) = \boldsymbol{a}(\gamma, \pi(y))$, μ-a.e.$(y), \forall \gamma \in \Gamma$. We have that $\boldsymbol{a} \sqsubseteq \boldsymbol{b} \implies \boldsymbol{a} \prec \boldsymbol{b}$. We also say that $\boldsymbol{a}, \boldsymbol{b}$ are *weakly isomorphic*, in symbols $\boldsymbol{a} \cong^w \boldsymbol{b}$, if $\boldsymbol{a} \sqsubseteq \boldsymbol{b}$ and $\boldsymbol{b} \sqsubseteq \boldsymbol{a}$. If $\boldsymbol{a} \sqsubseteq \boldsymbol{b}$ and π is as above, then π is a Borel homomorphism from $\boldsymbol{G}(S, \boldsymbol{b})$ to $\boldsymbol{G}(S, \boldsymbol{a})$ almost everywhere, so we have:

Proposition 6.45. *Let* $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ *and* $\boldsymbol{b} \in \mathrm{FR}(\Gamma, Y, \nu)$*. Then*

$$\boldsymbol{a} \sqsubseteq \boldsymbol{b} \implies \chi_\mu(\boldsymbol{G}(S, \boldsymbol{a})) \geq \chi_\nu(\boldsymbol{G}(S, \boldsymbol{b})).$$

Thus $\chi_\mu(\boldsymbol{G}(S, \boldsymbol{a}))$ *is an invariant of weak isomorphism.*

Now consider a standard Borel space X and let μ be a Borel probability measure on X not supported by a single point. Let μ^Γ be the product measure on X^Γ which is supported on $F(X^\Gamma)$. The action of Γ on $(F(X^\Gamma), \mu^\Gamma)$ is called a *Bernoulli shift*. By [AW] all Bernoulli shifts of an infinite group Γ are weakly equivalent, thus we have the following result:

Theorem 6.46. *Let (Γ, S) be an infinite marked group. Then for any standard Borel space X, any Borel probability measure μ on X, not supported by a single point, and for λ the Lebesgue measure on $[0,1]$, we have*

$$i_{\mu^\Gamma}(\boldsymbol{G}(S, X^\Gamma)) = i_{\lambda^\Gamma}(\boldsymbol{G}_\infty(\Gamma, S)).$$

and

$$\chi^{ap}_{\mu^\Gamma}(\boldsymbol{G}(S, X^\Gamma)) = \chi^{ap}_{\lambda^\Gamma}(\boldsymbol{G}_\infty(\Gamma, S)).$$

Bowen [B] shows that if $\Gamma \geq \mathbb{F}_2$ all Bernoulli shifts are actually weakly isomorphic. Therefore the following holds:

Theorem 6.47. *Let (Γ, S) be marked group with $\Gamma \geq \mathbb{F}_2$. Then for any standard Borel space X, any Borel probability measure μ on X, not supported by a single point, and for λ the Lebesgue measure on $[0,1]$, we have*

$$\chi_{\mu^\Gamma}(\boldsymbol{G}(S, X^\Gamma)) = \chi_{\lambda^\Gamma}(\boldsymbol{G}_\infty(\Gamma, S)).$$

Ball [Ba] showed that if (Γ, S) is a non-amenable marked group and for each finite $n \geq 2$ we let μ_n be the uniform measure on n, then there is some $n = n(\Gamma)$ such that any Bernoulli shift of Γ is a factor of the Bernoulli shift on (n^Γ, μ_n^Γ). Also if Γ has infinitely may ends, then $n(\Gamma)$ can be taken to be 2. Such results clearly fail for amenable groups by entropy considerations.

We will next discuss results concerning the independence number i_μ in connection with properties of the group Γ. Below for any graph $\boldsymbol{G}$, we denote by $g_{odd}(\boldsymbol{G})$ the *odd girth of* $\boldsymbol{G}$, i.e., the minimum size of an odd length cycle in $\boldsymbol{G}$, if $\boldsymbol{G}$ is not bipartite, and we let $g_{odd}(\boldsymbol{G}) = \infty$ otherwise.

Proposition 6.48. *For any infinite marked group (Γ, S), with $|S| = d, g = g_{odd}(\mathbf{Cay}(\Gamma, S))$, and any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$), we have*

$$\frac{1}{d} \leq i_\mu(\boldsymbol{G}(S, \boldsymbol{a})) \leq \frac{1}{2} - \frac{1}{2g}.$$

These bounds follow from Remark 6.37 and [CK, 4.5]. In particular, if $\mathbf{Cay}(\Gamma, S)$ is bipartite, then $g = \infty$, so we have

$$i_\mu(\boldsymbol{G}(S, \boldsymbol{a})) \leq \frac{1}{2},$$

for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$. (Note that $\mathbf{Cay}(\Gamma, S)$ is bipartite iff there is a homomorphism φ from Γ to $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ sending S to 1.) In fact we have the following characterization:

Theorem 6.49 ([CK, 4.6])**.** *Let (Γ, S) be an infinite marked group. Then the following are equivalent:*
(i) $\mathbf{Cay}(\Gamma, S)$ *is bipartite,*
(ii) There is $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ *with* $i_\mu(\boldsymbol{G}(S, \boldsymbol{a})) = \frac{1}{2}$.
(iii) There is an ergodic $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ *with* $i_\mu(\boldsymbol{G}(S, \boldsymbol{a})) = \frac{1}{2}$.

Consider now free ergodic actions, whose set we will denote by the symbol $\mathrm{FRERG}(\Gamma, X, \mu)$. This is again a G_δ set and thus a Polish space. There is a $\prec$-maximum element $\boldsymbol{a}^{erg}_{\Gamma,\infty}$ in the space $\mathrm{FRERG}(\Gamma, X, \mu)$ and $\boldsymbol{a}_{\Gamma,0}$ is the $\prec$-minimum element of $\mathrm{FRERG}(\Gamma, X, \mu)$, and it is shown in [CK, page 155] that we have:

$$\{i_\mu(\boldsymbol{G}(S, \boldsymbol{a}))\colon \boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)\} = [i_\mu(\boldsymbol{G}(S, \boldsymbol{a}_{\Gamma,0})), i_\mu(\boldsymbol{G}(S, \boldsymbol{a}^{erg}_{\Gamma,\infty}))].$$

However the structure of the set $\{i_\mu(\boldsymbol{G}(\Gamma, S, \boldsymbol{a}))\colon \boldsymbol{a} \in \mathrm{FRERG}(\Gamma, X, \mu))\}$ is not well understood and may very well depend on the structure of the group Γ. In [CKT-D, 9.1] it is shown, using ultraproduct techniques, that if Γ has Kazhdan's property (T), then this set is closed.

For the shift action of the free group $\mathbb{F}_n$ we have the following upper bound:

Theorem 6.50. *Let $\mathbb{F}_n$ be the free group with $n \geq 1$ generators and let $S = \{\gamma_1^{\pm 1}, \ldots, \gamma_n^{\pm 1}\}$, with $\{\gamma_1, \ldots, \gamma_n\}$ a free set of generators. Let μ be the usual product measure on $[0,1]^{\mathbb{F}_n}$. Then*

$$i_\mu(\boldsymbol{G}_\infty(\mathbb{F}_n, S)) \leq \min\left(\tfrac{\sqrt{2n-1}}{n+\sqrt{2n-1}}, \tfrac{\log 2n}{n}\right).$$

The proof of the upper bound $\frac{\sqrt{2n-1}}{n+\sqrt{2n-1}}$ in Theorem 6.50, which is due to [CK, 4.17], is based on first showing an analog of the so-called Hoffman bound in finite graphs, which establishes a connection between the independence number and the norm of the averaging operator associated to the Koopman unitary representation of the shift action. This analog was also independently proved in [LN]. One then applies a result of Kesten which gives a bound for the norm of this operator.

The upper bound $\frac{\log 2n}{n}$, which is better if $n \geq 5$, is proved by totally different methods, using results on random regular graphs and the ultraproduct method. We sketch the proof below.

Using the theory of random regular graphs, it follows that there is a sequence $\boldsymbol{G}_m = (X_m, G_m)$ of $2n$-regular finite graphs such that:

(i) $|X_m|$ is even and $|X_m| \to \infty$ as $m \to \infty$,

(ii) There is an action a_m of $\mathbb{F}_m$ on X_m which generates the graph $\boldsymbol{G}_m$ (i.e., two points are connected by an edge if a generator in S sends one of the points to the other),

(iii) $i(\boldsymbol{G}_m) \leq \frac{\log 2n}{n}$, where $i(\boldsymbol{G}_m)$ is the *independence ratio* of $\boldsymbol{G}_m$, i.e., the independence number of $\boldsymbol{G}_m$ with respect to the normalized counting measure μ_m on X_m,

(iv) For each $r \geq 1$ and $x \in X_m$, denote by $B_r^{\boldsymbol{G}_m}(x)$ the ball of radius r around x, i.e., the set of points in X_m whose $\boldsymbol{G}_m$-distance from x is at most r. Denote also by $\boldsymbol{B}_r^{\boldsymbol{G}_m}(x)$ the induced subgraph $\boldsymbol{G}_m | B_r^{\boldsymbol{G}_m}(x)$. Similarly denote by $\boldsymbol{B}_r^n$ the induced subgraph of $\mathbf{Cay}(\mathbb{F}_n, S)$ on the ball of radius r around the identity $1 \in \mathbb{F}_n$. Then we have

$$\frac{|\{x \in X_m \colon (\boldsymbol{B}_r^{\boldsymbol{G}_m}(x), x) \cong (\boldsymbol{B}_r^n, 1)\}|}{|X_m|} \to 1$$

as $m \to \infty$. Here $(\boldsymbol{B}_r^{\boldsymbol{G}_m}(x), x) \cong (\boldsymbol{B}_r^n, 1)$ means that there is an isomorphism of $\boldsymbol{B}_r^{\boldsymbol{G}_m}(x)$ with $\boldsymbol{B}_r^n$ that sends x to 1.

See [RW, page 4] for (ii), [Bol, Corollary 3] for (iii) and [W, 2.7] for (iv).

Below we use the notation and terminology in [CKT-D] concerning ultraproducts. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and denote by $X_{\mathcal{U}}, \boldsymbol{G}_{\mathcal{U}}, \mu_{\mathcal{U}}, a_{\mathcal{U}}$ the ultraproducts, resp., of $X_m, \boldsymbol{G}_m, \mu_m, a_m$. Thus $a_{\mathcal{U}}$ acts freely $\mu_{\mathcal{U}}$-a.e., and generates $\boldsymbol{G}_{\mathcal{U}}$. Moreover $i_{\mu_{\mathcal{U}}}(\boldsymbol{G}_{\mathcal{U}}) = \lim_{m \to \mathcal{U}} i(\boldsymbol{G}_m) \leq \frac{\log 2n}{n}$. Now, as in [CKT-D, Section 5], let $\boldsymbol{a} \in \mathrm{FR}(\mathbb{F}_n, X, \mu)$ be a factor of $a_{\mathcal{U}}$. Then $i_\mu(\boldsymbol{G}(S, \boldsymbol{a})) \leq i_{\mu_{\mathcal{U}}}(\boldsymbol{G}_{\mathcal{U}}) \leq \frac{\log 2n}{n}$. But by [AW], the shift action of $\mathbb{F}_n$ on $[0,1]^{\mathbb{F}_n}$ is weakly contained in $\boldsymbol{a}$, so $i_\mu(\boldsymbol{G}_\infty(\mathbb{F}_n, S)) \leq \frac{\log 2n}{n}$.

On the other hand, from results of Lauer-Wormald [LW] (see also [RV, Section 2]) it follows that for $n \geq 2$,

$$i_\mu(\boldsymbol{G}_\infty(\mathbb{F}_n, S)) \geq \frac{1}{2}(1 - (2n-1)^{-\frac{1}{n-1}}),$$

which is asymptotically equal to $\frac{\log 2n}{2n}$. Thus asymptotically $i_\mu(\boldsymbol{G}_\infty(\mathbb{F}_n, S))$ is within a factor of $\frac{1}{2}$ of the upper bound $\frac{\log 2n}{n}$.

Finally it is shown in [CK, 4.18] that $i_\mu(\boldsymbol{G}_\infty(\mathbb{F}_n, S)) \geq i_\mu(\boldsymbol{G}_\infty(\mathbb{F}_{n+1}, S))$ from which it is derived that for infinitely many n, there are at least three distinct values of $i_\mu(\boldsymbol{G}(S, \boldsymbol{a}))$, for $\boldsymbol{a} \in \mathrm{FRERG}(\mathbb{F}_n, X, \mu)$. In [CKT-D, 9.2] examples of (Γ, S) are found for which $\{i_\mu(\boldsymbol{G}(S,\boldsymbol{a}))\colon \boldsymbol{a} \in \mathrm{FRERG}(\Gamma, X, \mu))\}$ is infinite.

On the other extreme, for amenable groups we have the following result, where we define the independence number, in symbols $i(\Gamma, S)$, of the Cayley graph $\mathbf{Cay}(\Gamma, S)$ for an amenable group Γ, by

$$i(\Gamma, S) = \lim_{n\to\infty} i(F_n, S),$$

where (F_n) is a Følner sequence for Γ and $i(F_n, S)$ is the independence ratio of the finite induced subgraph $\mathbf{Cay}(\Gamma, S)|F_n$.

Theorem 6.51 ([CK, 4.10]). *If Γ is an infinite amenable group, then we have that $i_\mu(\boldsymbol{G}(S,\boldsymbol{a})) = i(\Gamma, S)$, for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$.*

In particular, if $\mathbf{Cay}(\Gamma, S)$ is bipartite, then $i_\mu(\boldsymbol{G}(S,\boldsymbol{a})) = \frac{1}{2}$, for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$. In fact when $\mathbf{Cay}(\Gamma, S)$ is bipartite, one can characterize various properties of the group Γ in terms of the behavior of the parameter $i_\mu(\boldsymbol{G}(S,\boldsymbol{a}))$, for $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$.

Theorem 6.52 ([CK, 4.14, 4.15]). *Let (Γ, S) be an infinite marked group with $\mathbf{Cay}(\Gamma, S)$ bipartite. Then we have:*

(i) Γ is amenable iff $i_\mu(\boldsymbol{G}(S,\boldsymbol{a}))$ is constant (and equal to $\frac{1}{2}$) for every $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$.

(ii) Γ has property (T) *iff $i_\mu(\boldsymbol{G}(S,\boldsymbol{a})) < \frac{1}{2}$ for every $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, which is weak mixing.*

(ii) Γ does not have the Haagerup Approximation Property (HAP) *iff*

$$i_\mu(\boldsymbol{G}(S,\boldsymbol{a})) < \frac{1}{2},$$

for every $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, which is mixing.

Finally, let us note that in the definition of the parameter

$$i_\mu(\boldsymbol{G}(S,\boldsymbol{a})) = \sup\{\mu(A)\colon A \text{ is a Borel independent set in } \boldsymbol{G}(S,\boldsymbol{a})\},$$

the supremum may not be attained (see e.g., [CK, page 148]). However, we have the following result, which is proved again using ultraproduct methods.

Theorem 6.53 ([CKT-D, Theorem 2]). *Let (Γ, S) be an infinite marked group. Then for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, there is $\boldsymbol{b} \in \mathrm{FR}(\Gamma, X, \mu)$ with $\boldsymbol{a} \sim \boldsymbol{b}$, so that $i_\mu(\boldsymbol{G}(S, \boldsymbol{a})) = i_\mu(\boldsymbol{G}(S, \boldsymbol{b}))$, and moreover the supremum in $i_\mu(\boldsymbol{G}(S, \boldsymbol{b}))$ is attained.*

We next discuss approximate and measurable chromatic numbers for graphs associated to measure preserving group actions. Given (Γ, S) with $|S| = d$, we have

$$2 \leq \chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a}_{\Gamma,0})) \leq \chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a})) \leq \chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a}_{\Gamma,\infty})) \leq d,$$

for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, by Remark 6.8. On the other hand

$$2 \leq \chi_\mu(\boldsymbol{G}(S, \boldsymbol{a})) \leq \max\{3, d\},$$

for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, by Theorem 6.36. If $\Gamma = \mathbb{Z}, S = \{\pm 1\}$ and $\boldsymbol{a}$ is the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$, then $\chi_\mu(\boldsymbol{G}(S, \boldsymbol{a})) = 3$ and $\chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a})) = 2$. The following problem was raised in [CK, page 160]:

Problem 6.54. *Are there $\Gamma, S, \boldsymbol{a}$ with $\chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a})) + 1 < \chi_\mu(\boldsymbol{G}(S, \boldsymbol{a}))$?*

Concerning the range of the functions $\chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a})), \chi_\mu(\boldsymbol{G}(S, \boldsymbol{a}))$ for $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, we have the following result, proved again by ultraproduct methods.

Theorem 6.55 ([CKT-D, Theorem 2]). *Let (Γ, S) be an infinite marked group. Then for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, there is $\boldsymbol{b} \in \mathrm{FR}(\Gamma, X, \mu)$ with $\boldsymbol{a} \sim \boldsymbol{b}$ and $\chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a})) = \chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{b})) = \chi_\mu(\boldsymbol{G}(S, \boldsymbol{b}))$*

Thus the range of χ_μ^{ap} is contained in that of χ_μ. On the other hand, we do not know in general if χ_μ^{ap}, χ_μ can take every value in the interval of integers determined by the above lower and upper bounds (for a fixed (Γ, S)).

Concerning bipartite graphs, we have an analog of the characterization in Theorem 6.49.

Theorem 6.56 ([CK, 4.6]). *Let (Γ, S) be an infinite marked group. Then the following are equivalent:*

(i) $\mathbf{Cay}(\Gamma, S)$ is bipartite,

(ii) There is $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ with $\chi_\mu(\boldsymbol{G}(S, \boldsymbol{a})) = 2$.

(iii) There is an ergodic $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ with $\chi_\mu(\boldsymbol{G}(S, \boldsymbol{a})) = 2$.

As in the case of independence numbers, very little is known in general about the range of the functions χ_μ^{ap}, χ_μ for ergodic actions.

In the case of amenable groups the following result is a consequence of Proposition 6.21.

Theorem 6.57 ([CK, 4.7])**.** *Let (Γ, S) be an infinite marked group with Γ amenable. Then for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$,*

$$\chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a})) = \chi(\mathbf{Cay}(\Gamma, S))$$

Thus if $\mathbf{Cay}(\Gamma, S))$ is also bipartite, $\chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a})) = 2$, for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$.

Analogously to Theorem 6.52, we can characterize properties of the group Γ in terms of the function χ_μ^{ap}.

Theorem 6.58 ([CK, 4.14, 4.15])**.** *Let (Γ, S) be an infinite marked group with $\mathbf{Cay}(\Gamma, S)$ bipartite. Then we have:*

(i) Γ is amenable iff $\chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a}))$ is constant (and equal to 2) for every $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$.

(ii) Γ has property (T) *iff $\chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a})) \geq 3$ for every $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, which is weak mixing.*

(ii) Γ does not have the Haagerup Approximation Property (HAP) *iff*

$$\chi_\mu^{ap}(\boldsymbol{G}(S, \boldsymbol{a})) \geq 3,$$

for every $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, which is mixing.

Finally, we have the following bound for the shift action of $\mathbb{F}_n$, which is an immediate consequence of Theorem 6.50.

Theorem 6.59. *Let $\mathbb{F}_n$ be the free group with $n \geq 1$ generators and let$S = \{\gamma_1^{\pm 1}, \dots, \gamma_n^{\pm 1}\}$, with $\{\gamma_1, \dots, \gamma_n\}$ a free set of generators. Let μ be the usual product measure on $[0,1]^{\mathbb{F}_n}$. Then*

$$\chi_\mu(\boldsymbol{G}_\infty(\mathbb{F}_n, S)) \geq \chi_\mu^{ap}(\boldsymbol{G}_\infty(\mathbb{F}_n, S)) \geq \max(3, \tfrac{n}{\log 2n}).$$

Strengthening Theorem 6.9 in the case of the graphs arising from the shift action of a group, Bernshteyn has proved the following upper bound on $\chi_\mu(\boldsymbol{G}_\infty(\Gamma, S))$, which significantly improves the previously known best upper bound coming from Theorem 6.7:

Theorem 6.60 ([Ber, Theorem 1.1]). *Suppose* (Γ, S) *is a marked group with* $|S| = d$*, and* μ *is the usual product measure on* $[0,1]^{\mu}$*. If* $g(\boldsymbol{G}_\infty(\Gamma, S)) \geq 4$*, then*

$$\chi_\mu(\boldsymbol{G}_\infty(\Gamma, S)) = O\left(\frac{d}{\log d}\right)$$

and if $g(\boldsymbol{G}_\infty(\Gamma, S)) \geq 5$*, then*

$$\chi_\mu(\boldsymbol{G}_\infty(\Gamma, S)) = (1 + o(1))\frac{d}{\log d}.$$

In the case where $\Gamma = \mathbb{F}_n$ with its usual set of generators, we therefore have asymptotically matching upper and lower bounds which differ only by a factor of two:

$$\frac{n}{\log 2n} \leq \chi_\mu^{ap}(\boldsymbol{G}_\infty(\mathbb{F}_n, S)) \leq \chi_\mu(\boldsymbol{G}_\infty(\mathbb{F}_n, S)) \leq (1 + o(1))\frac{2n}{\log 2n}$$

It remains an open problem to determine the exact growth rate of these chromatic numbers:

Problem 6.61. *Compute* $\chi_\mu(\boldsymbol{G}_\infty(\mathbb{F}_n, S))$ *and* $\chi_\mu^{ap}(\boldsymbol{G}_\infty(\mathbb{F}_n, S))$*.*

Finally we note an application of Theorem 6.50 to finite graphs. Using probabilistic methods, Erdős showed that there are finite graphs that have simultaneously arbitrarily large girth and arbitrarily small independence ratio (and thus arbitrarily large chromatic number). These requirements are in apparent conflict since large girth or even large odd girth signifies approximation to being bipartite thus having chromatic number 2. (Recall that the independence ratio of a finite graph is the ratio of the maximum size of an independent set divided by the number of vertices.)

It is highly non-trivial to produce *explicit* families of finite graphs with arbitrarily large girth and arbitrarily small independence ratio. Lubotzky-Phillips-Sarnak [LPS] produced explicit examples (that are actually *Ramanujan graphs*) using deep results from number theory.

It is a bit easier to produce explicit families that have arbitrarily large odd girth and arbitrarily small independence ratio. One example (pointed out to us by Sudakov) of such a family consists of the so-called *Borsuk graphs*: Consider the set $\{0,1\}^n$ and the graph $B(\epsilon, n)$ on this set, where two points x, y are connected iff the Hamming distance between x and the flip of y is $< \epsilon$. By making $\epsilon \to 0$ and $n \to \infty$, these graphs have arbitrarily large odd girth and arbitrarily small independence ratio. One can prove this using concentration of measure in $\{0,1\}^n$, with the (normalized) counting measure and Hamming distance.

Theorem 6.50 about the shift action of $\mathbb{F}_n$ can be also used to produce another such family. Let $\Gamma = \mathbb{F}_n$ and S be as in that theorem and fix an increasing sequence of finite subsets K_m of Γ covering it. Consider the shift action of Γ on 2^Γ and for each $p \in 2^{K_m}$, consider the basic open set $N_p = \{f \in 2^\Gamma : f|K_m = p\}$. Then define a graph on 2^{K_m} by connecting p, q by an edge if an element of S moves by the action N_p so that it intersects N_q. Let $\boldsymbol{G}_{n,k,m}$ be the induced subgraph restricted to vertices that do not belong to odd cycles of size $\leq k$. Then we have:

Theorem 6.62 ([CK, 4.22])**.** *Given n, k, for all large enough m, depending on n, k, we have that the odd girth of $\boldsymbol{G}_{n,k,m}$ is bigger than k and the independence number is at most $i_\mu(\boldsymbol{G}_\infty(\mathbb{F}_n, S))$, so converges to 0 as $n \to \infty$.*

6.7 Invariant, random colorings of Cayley graphs

We discuss here some connections of the results of Section 5,**(F)** with problems in probability theory concerning random colorings of Cayley graphs.

Let (Γ, S) be an infinite marked group and $k \geq 1$. We denote by $\mathrm{Col}(\Gamma, S, k)$ the set of k-colorings of $\mathbf{Cay}(\Gamma, S)$. This is a closed subspace of the compact space k^Γ. The group Γ acts by shift on k^Γ and clearly $\mathrm{Col}(\Gamma, S, k)$ is invariant under this action. A Borel probability measure on $\mathrm{Col}(\Gamma, S, k)$ which is invariant under this action is called a *Γ-invariant, random k-coloring* of $\mathbf{Cay}(\Gamma, S)$.

The question of the existence of invariant, random colorings is discussed in Aldous-Lyons [AL, 10.5], where it is mentioned that Schramm (unpublished, 1997) had shown that if $|S| = d$, then there is a Γ-invariant, random $(d+1)$-coloring, a fact that is also a consequence of Proposition 5.4. Also [AL, 10.5] note that if Γ is sofic, then there is a Γ-invariant, random d-coloring. It was finally shown in [CKT-D, 7.4] that for *any* (Γ, S) with $|S| = d$, there is a Γ-invariant, random d-coloring of $\mathbf{Cay}(\Gamma, S)$, i.e., a "random" version of Brooks' Theorem holds. This result was recently extended in various ways, that will be explained below, in the paper [CMT-D]. But first we will discuss the basic relationship between colorings of graphs induced by measure preserving actions and invariant, random colorings of Cayley graphs.

Let $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ and let $c\colon X \to k$ be a μ-measurable coloring of $\boldsymbol{G}(S, \boldsymbol{a})$. Then we define the μ-measurable map $F\colon X \to \mathrm{Col}(\Gamma, S, k)$ by $F(x)(\gamma) = c(\gamma^{-1} \cdot x)$. (As usual we will neglect here and below null sets; strictly speaking both c, F are only defined μ-a.e.) Then it is easy to check that F is equivariant with respect to $\boldsymbol{a}$ and the action of Γ on $\mathrm{Col}(\Gamma, S, k)$. Thus if $\nu = F_*\mu$, then ν is a Γ-invariant, random k-coloring of $\mathbf{Cay}(\Gamma, S)$, which is a *factor* of $\boldsymbol{a}$, with factor map F.

Conversely, if a Γ-invariant, random k-coloring ν of $\mathbf{Cay}(\Gamma, S)$ is a factor of an action $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ via a factor map F (i.e., F is equivariant and sends μ to ν), then we can define a μ-measurable coloring $c\colon X \to k$ by $c(x) = f(x)(1)$. Clearly these two processes are inverses of each other.

Moreover, given a Γ-invariant, random k-coloring ν of $\mathbf{Cay}(\Gamma, S)$, we can construct $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, $c\colon X \to k$ a μ-measurable coloring of $\boldsymbol{G}(S, \boldsymbol{a})$, and a factor map F which gives ν as above. Indeed, using a trick of Lyons, let $\boldsymbol{b} \in \mathrm{FR}(\Gamma, Y, \rho)$ and put $X = \mathrm{Col}(\Gamma, S, k) \times Y, \mu = \nu \times \rho$ and let $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ be the product of the action of Γ on $\mathrm{Col}(\Gamma, S, k)$ with the action $\boldsymbol{b}$ (the only point in taking this product is to make sure that $\boldsymbol{a}$ is free). Finally, define $c\colon X \to k$ by $c(p, y) = p(0)$. This is μ-measurable and a coloring since if $(q, z) = \boldsymbol{a}(s, (p, y))$, for $s \in S$, then $c(q, z) = q(0) = p(s^{-1}) \neq p(0) = c(p, y)$.

The combination of Theorem 6.55 and Remark 6.37, shows that for any (Γ, S), with $|S| = d$, there is a Γ-invariant, random d-coloring of $\mathbf{Cay}(\Gamma, S)$, see [CKT-D, 7.4]. This has been now substantially strengthened to the following, which is a corollary of Theorem 6.36, when $d \geq 3$:

Theorem 6.63 ([CMT-D]). *Let (Γ, S) be an infinite marked group with $|S| = d \geq 3$. Then for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, there is a Γ-invariant, random d-coloring of $\mathbf{Cay}(\Gamma, S)$, which is a factor of $\boldsymbol{a}$.*

Remark 6.64. This stronger statement fails for $d = 2$. By Remark 6.30, for any infinite marked group (Γ, S), there is no *weak mixing* Γ-invariant, random 2-coloring of $\mathbf{Cay}(\Gamma, S)$, thus no such random coloring can be a factor of any weak mixing action. On the other hand, for every (Γ, S) with bipartite $\mathbf{Cay}(\Gamma, S)$, there are exactly two elements of $\mathrm{Col}(\Gamma, S, 2)$, so the uniform measure is ergodic Γ-invariant.

In particular, Theorem 6.63 shows that for any (Γ, S) with $|S| = d \geq 3$, there is a Γ-invariant, random d-coloring of $\mathbf{Cay}(\Gamma, S)$, which is a factor of the shift action of Γ on $[0,1]^\Gamma$ with the usual product measure. Such random colorings are called *factors of* IID in probability theory.

We note here that, by Theorem 6.59, we have that any $\mathbb{F}_n$-invariant, random k-coloring of $\mathbf{Cay}(\mathbb{F}_n, S)$, with the usual set of generators S, which is a factor of IID, must have $k \geq \max(3, \frac{n}{\log 2n})$. Finally in [LN, Section 5] it is pointed out that there is a $\mathbb{F}_n$-invariant, random 3-coloring of $\mathbf{Cay}(\mathbb{F}_n, S)$, which is *mixing*.

There is a stronger notion of invariance for random colorings of Cayley graphs. Let $\mathrm{Aut}_{\Gamma,S} = \mathrm{Aut}(\mathbf{Cay}(\Gamma, S))$ be the automorphism group of $\mathbf{Cay}(\Gamma, S)$ with the pointwise convergence topology, in which it is a Polish locally compact group. Identifying $\gamma \in \Gamma$ with the left-translation map $\delta \to \gamma\delta$, we view Γ as a closed subgroup of $\mathrm{Aut}_{\Gamma,S}$. Clearly $\mathrm{Aut}_{\Gamma,S}$ also acts on any X^Γ by $\varphi \cdot p(\gamma) = p(\varphi^{-1}(\gamma))$ and again $\mathrm{Col}(\Gamma, S, k)$ is invariant under this action (when viewed as a closed subspace of k^Γ). We call a Borel probability measure on $\mathrm{Col}(\Gamma, S, k)$ which is invariant under this action an $\mathrm{Aut}_{\Gamma,S}$*-invariant, random* k*-coloring* of $\mathbf{Cay}(\Gamma, S)$. If such random coloring is a factor of the shift action of $\mathrm{Aut}_{\Gamma,S}$ on $[0,1]^\Gamma$, we again call it a $\mathrm{Aut}_{\Gamma,S}$*-factor of* IID.

It is a general fact that there is an $\mathrm{Aut}_{\Gamma,S}$-invariant, random k-coloring of $\mathbf{Cay}(\Gamma, S)$ iff there exists a Γ-invariant, random k-coloring of $\mathbf{Cay}(\Gamma, S)$; see [CKT-D, 7.6]. From this it follows that a "random" version of Problem 6.38 has a positive answer: If $(\Gamma, S), (\Delta, T)$ are two infinite marked groups with isomorphic Cayley graphs, then for every k, there is a Γ-invariant, random k-coloring of $\mathbf{Cay}(\Gamma, S)$ iff there is a Δ-invariant, random k-coloring of $\mathbf{Cay}(\Delta, T)$.

It also follows that for any (Γ, S) with $|S| = d$, there exists an $\mathrm{Aut}_{\Gamma,S}$-invariant, random d-coloring of $\mathbf{Cay}(\Gamma, S)$. In [CKT-D, 7.7] it was shown that if in addition Γ has finitely many ends but is not isomorphic to either $\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$, then such random colorings can be found that are $\mathrm{Aut}_{\Gamma,S}$-factors of IID. This was recently extended to all (Γ, S) modulo these two exceptions.

Theorem 6.65 ([CMT-D, Corollary 5.4]). *Let* (Γ, S) *be an infinite marked group with* $|S| = d$ *such that* Γ *is not isomorphic to either* $\mathbb{Z}$ *or* $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$. *Then there is a* $\mathrm{Aut}_{\Gamma,S}$*-invariant, random* d*-coloring of* $\mathbf{Cay}(\Gamma, S)$ *which is a* $\mathrm{Aut}_{\Gamma,S}$*-factor of* IID.

The following question was raised by Lyons and Schramm (unpublished, 1997 – see also [LN, Section 5]):

Problem 6.66. *Let* (Γ, S) *be an infinite marked group. If* $\chi = \chi(\mathbf{Cay}(\Gamma, S))$*, is there a* Γ*-invariant, random* χ*-coloring of* $\mathbf{Cay}(\Gamma, S)$*?*

Since the space $\mathrm{Col}(\Gamma, S, \chi)$ is nonempty, compact and Γ acts continuously on it, clearly such a random coloring exists if Γ is amenable.

Remark 6.67. If $\boldsymbol{G} = (X, G)$ is a countable graph, one can also define as before the compact space $\mathrm{Col}(\boldsymbol{G}, k)$ of k-colorings of $\boldsymbol{G}$ and similarly define $\mathrm{Aut}(\boldsymbol{G})$-invariant, random k-colorings of $\boldsymbol{G}$. In the recent work of Agol [Ag] (see also [Bes, Proposition 8.2]) the following coloring lemma was proved: If $\Delta(\boldsymbol{G}) \leq d$, then there exists an $\mathrm{Aut}(\boldsymbol{G})$-invariant, random $(d+1)$-coloring of $\boldsymbol{G}$ (compare this with Proposition 5.4 and the paragraph preceding Theorem 6.65).

We conclude with some remarks and questions concerning invariant, random edge colorings. For an infinite marked group (Γ, S), we define $\mathrm{Ecol}(\Gamma, S, k)$ to be the space of edge k-colorings of $\mathbf{Cay}(\Gamma, S)$. This is a closed subspace of the compact space $k^{E(\Gamma,S)}$, where $E(\Gamma, S)$ is the set of edges of $\mathbf{Cay}(\Gamma, S)$ (where an edge is viewed here as an unordered pair). Again Γ and $\mathrm{Aut}_{\Gamma,S}$ act by shift on $k^{E(\Gamma,S)}$ and $\mathrm{Ecol}(\Gamma, S, k)$ is invariant under this action. A Borel probability measure on $\mathrm{Ecol}(\Gamma, S, k)$ invariant under the Γ-action (resp., the $\mathrm{Aut}_{\Gamma,S}$-action) is called a Γ*-invariant* (resp., $\mathrm{Aut}_{\Gamma,S}$*-invariant) random edge* k*-coloring* of $\mathbf{Cay}(\Gamma, S)$. We similarly define what it means for such random edge colorings to be factors of IID.

The basic connections between measurable colorings of graphs $\boldsymbol{G}(S, \boldsymbol{a})$ and Γ-invariant, random colorings of $\mathbf{Cay}(\Gamma, S)$ carry over to the present context of edge colorings, mutatis mutandis. In particular, by the paragraph following Problem 6.39, there is an $\mathbb{F}_n$-invariant, random edge $(2n)$-coloring of $\mathbf{Cay}(\mathbb{F}_n, S)$, where S is the usual set of generators. However it is not known if there is a $\mathbb{F}_n$-invariant, random edge $(2n)$-coloring of $\mathbf{Cay}(\mathbb{F}_n, S)$ which is a factor of IID. Similarly there is a $\mathrm{Aut}_{\Delta_d,T}$-invariant, random edge d-coloring of $\mathbf{Cay}(\Delta_d, T)$, where Δ_d, T are defined as in the second paragraph following Problem 6.39. In fact Lyons [L, §2] shows that there is a unique such random edge coloring. Moreover for each action $\boldsymbol{a} \in \mathrm{FR}(\Delta_d, X, \mu)$, there is a Δ_d-invariant, random edge d-coloring of $\mathbf{Cay}(\Delta_d, T)$ which is a factor of $\boldsymbol{a}$ but Lyons [L, 2.5] raises the question of whether the unique $\mathrm{Aut}_{\Delta_d,T}$-invariant, random edge d-coloring of $\mathbf{Cay}(\Delta_d, T)$ is a $\mathrm{Aut}_{\Delta_d,T}$-factor of IID, when $d \geq 3$ (this clearly fails for $d = 2$ by the paragraph following Problem 6.38).

7. Possible chromatic numbers

In this section, we discuss what relationships exist in general between the various chromatic numbers we have considered above. Graphs with no odd cycles (and especially acyclic graphs) play an important role here since such graphs are those for which $\chi \leq 2$. To begin, we note that, individually, the chromatic numbers we have been discussing can take any value even when restricted to the class of acyclic Borel graphs. Hence, there is no relationship between χ and these other chromatic numbers, except the obvious relation that χ is less than or equal to them, since we can take disjoint unions of these graphs with complete graphs.

Theorem 7.1 ([M08, Theorem 3.9], [KST, Appendix], [CK])**.** *For each of the chromatic numbers χ_B, χ_{BM}, χ_μ, and χ_M and any value in*

$$\{1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}\},$$

there is an acyclic Borel graph whose given chromatic number achieves this value.

In fact, the results we refer to for χ_{BM} and χ_μ are slightly stronger: these graphs have the property that they achieve the given value for χ_{BM} or χ_μ on any comeager or conull Borel set, respectively. Hence, the result for χ_M follows from the result from χ_μ by constructing a new graph that is the restriction of a graph with a given value κ for χ_μ to a conull Borel set on which it has a Borel κ-coloring, so the resulting graph has $\kappa = \chi_\mu \leq \chi_M \leq \chi_B = \kappa$.

In Section 2, we noted that $\chi \leq \chi_\mu \leq \chi_B$ and $\chi \leq \chi_{BM} \leq \chi_B$. Using the above result, we see that in general there are no other relationships between these chromatic numbers for Borel graphs.

Corollary 7.2. *For any possible way of assigning values from the set*

$$\{1, 2, 3, \dots, \aleph_0, 2^{\aleph_0}\}$$

to χ, χ_μ, χ_{BM}, and χ_B that is consistent with the inequalities $\chi \leq \chi_\mu \leq \chi_B$ and $\chi \leq \chi_{BM} \leq \chi_B$, there is a Borel graph $\boldsymbol{G}$ on a Polish space X with a probability measure μ realizing these four values.

Proof. Let $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, $\boldsymbol{G}_3$ be acyclic Borel graphs where $\chi_{BM}(\boldsymbol{G}_1)$, $\chi_\mu(\boldsymbol{G}_2)$, and $\chi_B(\boldsymbol{G}_3)$ achieve our desired values for χ_{BM}, χ_μ, and χ_B. We may assume $\chi_{BM}(\boldsymbol{G}_1) = \chi_B(\boldsymbol{G}_1)$ and $\chi_\mu(\boldsymbol{G}_2) = \chi_B(\boldsymbol{G}_2)$ by restricting $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ to a Borel comeager or conull set, respectively, where their Baire measurable or μ-measurable colorings, respectively, become Borel, as discussed above. To obtain our desired graph we take the disjoint union of $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{G}_3$ along with a complete graph $\boldsymbol{G}_4$ having our desired value for χ. More precisely, to preserve our value of χ_{BM} from $\boldsymbol{G}_1 = (X_1, G_1)$, take an uncountable meager Borel subset $A \subseteq X_1$, partition A into two uncountable Borel pieces A_1 and A_2, and then using a Borel bijection between A and A_1, form an isomorphic copy of $\boldsymbol{G}_1$ on $X_1 \setminus A_2$, which therefore has the same value for χ_{BM} and χ_B. We can then arrange that in our disjoint union, $\boldsymbol{G}_2$, $\boldsymbol{G}_3$ and $\boldsymbol{G}_4$ are supported on the set A_2. $\square$

We may analogously ask what values χ, χ_μ, χ_{BM}, and χ_B can take on more restricted classes of Borel graphs, such as locally finite Borel graphs, d-regular Borel graphs, hyperfinite Borel graphs, etc. These are all open questions, though taking disjoint unions as in the proof above essentially reduces these problems to asking what pairs of values can be achieved between χ and each of the other chromatic numbers. For example, the possible pairs of values for χ and χ_{BM} among locally finite graphs are unknown, see Theorem 5.6 and Problem 6.19, as are the possible pairs of values for χ and χ_μ among d-regular Borel graphs (see Problem 6.61 for an interesting special case) and hyperfinite Borel graphs.

It is also interesting to consider the possible values of χ, χ_μ, χ_{BM}, and χ_B for more natural classes of graphs, as opposed to our ad-hoc disjoint unions above. For example, we can consider graphs of the form $\boldsymbol{G}(S, X^\Gamma)$ equipped with their natural product measures. Another natural class of graphs are the locally countable μ-measure preserving Borel graphs, which are *μ-ergodic* (i.e., all $E_{\boldsymbol{G}}$ invariant Borel sets are null or conull). The question was raised in [CK, Page 137] of what are the possible values of $k, l, m \in \{2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$ with $k \leq l \leq m$ for which there is such a $\boldsymbol{G}$ with $\chi(\boldsymbol{G}) = k, \chi_\mu(\boldsymbol{G}) = l, \chi_B(\boldsymbol{G}) = m$. By taking direct sums, it is clear that any such triple $k \leq l \leq m$ is possible, using the following two facts:

(i) There are locally countable acyclic Borel graphs whose Borel chromatic number takes any value m in the set $\{2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$; see Example 3.16 for $m = 2^{\aleph_0}$, Example 4.2 for $m = \aleph_0$ and Section 5, **(B)**, for finite m.

(ii) There are locally countable μ-measure preserving, ergodic Borel graphs whose μ-measurable chromatic number takes any value l in the set $\{2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$; see Example 3.15 for $l = 2^{\aleph_0}$, [CM1, Section 2] for $l = \aleph_0$ and [CK, Example 2.5], for finite l.

For edge chromatic numbers one may ask similar questions. Beyond the obvious facts that $\chi' \leq \chi'_\mu \leq \chi'_B$ and $\chi' \leq \chi'_{BM} \leq \chi'_B$, we have that if $\boldsymbol{G}$ is an analytic graph and $\chi'(\boldsymbol{G})$ is infinite, then $\chi'(\boldsymbol{G}) = \chi_B(\boldsymbol{G}) = \Delta(\boldsymbol{G})$ by Proposition 4.1 and Vizing's Theorem 6.10, and if $\chi'(\boldsymbol{G})$ is finite, then $\chi'_B(\boldsymbol{G}) \leq 2\chi'(\boldsymbol{G}) - 1$, by Proposition 5.4. Similarly to Corollary 7.2, one may take disjoint unions to essentially reduce this problem to the question of what relationships hold between the pairs χ' and χ'_μ and χ' and χ'_{BM}. These remain open problems (see the discussion after Problem 6.13).

Of course, the lack of any real relationship between χ_{BM} and χ_μ is unsurprising because of the orthogonality of measure and category. However, there are other pairs of chromatic numbers that do have much closer relationships, and here there are many interesting open questions. We mention a few.

Problem 7.3. *What pairs of values are possible for χ_μ and χ_μ^{ap} among acyclic Borel graphs?*

Indeed, we do not know of any acyclic Borel graphs $\boldsymbol{G}$ for which we have $\chi_\mu(\boldsymbol{G}) > \chi_\mu^{ap}(\boldsymbol{G}) + 1$ (see also Problem 6.54). We ask specifically about acyclic graphs here to sidestep the issue that χ_μ could be much larger than χ_μ^{ap} simply because χ could be large on a nullset.

Next, we consider χ_M and χ_B.

Problem 7.4. *What pairs of values are possible for χ_M and χ_B among Borel graphs?*

This is open when χ_B is finite but it is answered when χ_B is infinite. Note that $\chi_\mu(\boldsymbol{G}_0) = 3$ (for the usual product measure on $2^{\mathbb{N}}$), $\chi_B(\boldsymbol{G}_0) = 2^{\aleph_0}$; see the paragraph following Remark 4.17 and Example 4.16, resp. Also $\chi_M(\boldsymbol{G}_{S^\infty_{\mathbb{N}}}) = 3$ and $\chi_B(\boldsymbol{G}_{S^\infty_{\mathbb{N}}}) = \aleph_0$; see Example 5.2. Since for any $\kappa \in \{1, 2, 3 \ldots, \aleph_0, 2^{\aleph_0}\}$, there are acyclic Borel graphs with $\chi_M = \chi_B = \kappa$, by taking disjoint unions with $\boldsymbol{G}_0, \boldsymbol{G}_{S^\infty_{\mathbb{N}}}$ and complete graphs, for any $k \leq$

$l \leq m$, with $m \geq \aleph_0, l \geq 3$, we can find a Borel graph $\boldsymbol{G}$ with $\chi(\boldsymbol{G}) = k, \chi_M(\boldsymbol{G}) = l, \chi_B(\boldsymbol{G}) = m$. The assumption that $l \geq 3$ is required, since Theorem 4.32 implies that if $\chi_B > 2$, then $\chi_M > 2$.

Problem 7.4 essentially asks whether there is a dichotomy theorem showing that whenever χ_B is large and finite, there must be some measure μ with respect to which χ_μ is large. This problem is also related to Problem 5.24, since by Theorem 6.18 and Example 5.2, a sequence $(\boldsymbol{G}_n)$ as in Problem 5.24 must have $\chi_M(\boldsymbol{G}_n) = 3$, while $\chi_B(\boldsymbol{G}_n)$ must be finite and unbounded.

Similarly, we can ask analogous questions about the supremum of χ_{BM} over all Polish topologies compatible with the standard Borel structure on the space of a given graph. That is, supposing χ_B is large, must there be some Polish topology with respect to which χ_{BM} is large? Note that by Theorem 4.32, $\chi_B \geq 3 \implies \chi_{BM} \geq 3$, for some Polish topology, while the $\boldsymbol{G}_0$-dichotomy, see Section 3,**(D)**, shows that $\chi_B = 2^{\aleph_0} \implies \chi_{BM} = 2^{\aleph_0}$, for some Polish topology.

8. Other notions of coloring

Beyond vertex colorings and edge colorings, combinatorics studies a host of other coloring notions, many of which make sense in the descriptive setting. In this section we survey work that has been done on these other coloring concepts.

8.1 List coloring

Suppose $\boldsymbol{G} = (X, G)$ is a graph and L is a function mapping each $x \in X$ to a set $L(x)$. Then a *coloring of* $\boldsymbol{G}$ *from the lists* L is a coloring c of $\boldsymbol{G}$ that is a choice function for L; that is, for all $x \in X$ we have $c(x) \in L(x)$. Thus, the usual notion of a k-coloring corresponds to a coloring from the lists L where $L(x) = \{1, \ldots, k\}$ for all x. If $f\colon X \to \mathbb{N}$ is a function, then $\boldsymbol{G}$ is f*-list-colorable* or f*-choosable* if for every function L on X with $|L(x)| = f(x)$, $\boldsymbol{G}$ is has a coloring from the lists L. An important special case is when f takes the constant value k, and here we say $\boldsymbol{G}$ is k*-list-colorable* or k*-choosable*. The least k for which $\boldsymbol{G}$ is k-list-colorable is denoted $\mathrm{ch}(\boldsymbol{G})$.

In the descriptive setting, if $\boldsymbol{G} = (X, G)$ is a graph on a Polish space X, $f\colon X \to \mathbb{N}$ is a function and Γ is a class of functions between Polish spaces, then we say $\boldsymbol{G}$ is Γ f*-list-colorable* if for every Polish space Y and Borel function $L\colon X \to [Y]^{<\infty}$, with $|L(x)| = f(x)$, for all x, $\boldsymbol{G}$ has a Γ-coloring from the lists L. The least k for which $\boldsymbol{G}$ is Γ k-list-colorable is denoted $\mathrm{ch}_\Gamma(\boldsymbol{G})$.

An obvious greedy algorithm demonstrates that every graph $\boldsymbol{G}$ is f-list-colorable for the function $f(x) = \deg_{\boldsymbol{G}}(x) + 1$. Similarly, an obvious modification to Proposition 5.4 shows that every locally finite analytic graph is Borel f-list-colorable for $f(x) = \deg_{\boldsymbol{G}}(x) + 1$. Graphs that are f-list-colorable for the function $f(x) = \deg_{\boldsymbol{G}}(x)$ are said to be *degree-*

list-colorable. Borodin [Bor] and Erdős-Rubin-Taylor [ERT] independently characterized the degree-list-colorable graphs. Recall the notion of a Gallai tree from Section 5.

Theorem 8.1 ([Bor], [ERT], [CR, Section 8]). *A connected graph is degree-list-colorable iff it is not a finite Gallai tree.*

This can be viewed as a generalization of Brooks' theorem since a minimal counterexample to Brooks' theorem would be a regular graph, and the only regular Gallai trees are finite complete graphs and odd cycles. Conley, Marks and Tucker-Drob [CMT-D] have recently proved the following analogue of this result.

Theorem 8.2 ([CMT-D, Theorem 4.2]). *Suppose that $\boldsymbol{G} = (G, X)$ is a locally finite Borel graph on a Polish space X containing no connected components that are finite Gallai trees, and no infinite connected components that are 2-ended Gallai trees.*

(i) Let μ be any Borel probability measure on X. Then $\boldsymbol{G}$ is μ-measurably degree-list-colorable.

(ii) The graph $\boldsymbol{G}$ is Baire measurably degree-list-colorable.

The requirement that $\boldsymbol{G}$ does not contain 2-ended connected components is necessary since an easy ergodic theoretic or Baire category argument shows that for $\boldsymbol{G} = \boldsymbol{G}(S, 2^{\mathbb{Z}})$ (and μ the usual product measure on $2^{\mathbb{Z}}$) we have $\chi_\mu(\boldsymbol{G}) = \chi_{BM}(\boldsymbol{G}) = 3$; see paragraph following Proposition 6.21.

Similarly to the above, we can also define *list edge-coloring*, and its descriptive counterparts. If $\boldsymbol{G}$ is a graph, we let $\mathrm{ch}'(\boldsymbol{G})$ be the least k such that $\boldsymbol{G}$ is k-list-edge-colorable, and similarly for $\mathrm{ch}'_\Gamma(\boldsymbol{G})$. In this setting, the *list edge-coloring conjecture* is a well-known open problem.

Conjecture 8.3 ([JT, 12.20]). *If $\boldsymbol{G}$ is a finite graph, then $\mathrm{ch}'(\boldsymbol{G}) = \chi'(\boldsymbol{G})$.*

In the descriptive setting, the conjecture has a negative answer.

Proposition 8.4. *There is a 2-regular acyclic Borel graph $\boldsymbol{G}$ with $\chi'_B(\boldsymbol{G}) = 2$ and $\mathrm{ch}'_B(\boldsymbol{G}) = 3$.*

Proof. Consider the group $\Gamma = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle a_1, a_2 | a_1^2 = a_2^2 = 1\rangle$ with generators $S = \{a_1, a_2\}$, and let $\boldsymbol{G} = \boldsymbol{G}(S, 2^\Gamma)$. It is easy to see that there is no Borel function $f \subseteq G(S, 2^\Gamma)$ defined on the vertices of $\boldsymbol{G}$ such that

$\forall x(f^2(x) \neq x)$. Indeed, there is no such Borel function on any conull Borel set (with respect to the natural product measure μ) as one can see by using the fact that $x \mapsto a_1 \cdot x$ is measure-preserving and $x \mapsto a_1 a_2 \cdot x$ is ergodic, and similarly, there is no such Borel function on any comeager set by using the fact that $x \mapsto a_1 \cdot x$ is a homeomorphism and $x \mapsto a_1 a_2 \cdot x$ is generically ergodic. (See [M08b, Remark 4.6]). This is related to the fact that there is no measurable or Baire measurable choice of one end from each connected component of the graph $\boldsymbol{G}$.

Clearly $\chi'_B(\boldsymbol{G}) = 2$ by using the two generators in S as colors. Also since the line graph of $\boldsymbol{G}$ is 2-regular, $\boldsymbol{G}$ is Borel 3-list-edge-colorable (see the paragraph preceding Theorem 8.1). We show that $\boldsymbol{G}$ is not Borel 2-list-edge-colorable. The graph $\boldsymbol{G}$ has a Borel vertex coloring c with 3 colors by Proposition 5.4. Let L be the function given by $L(\{x, y\}) = \{c(x), c(y)\}$, which maps each edge to the color of its two incident vertices. Suppose c' is a Borel edge coloring of $\boldsymbol{G}$ from the lists L. This yields a Borel function f contradicting the above by setting $f(x) = y$ if y neighbors x and $c(x) = c'(\{x, y\})$. (Note that there is at most one x in each connected component without a corresponding y with $c(x) = c'(\{x, y\})$, and hence $f(x)$ is defined on a comeager/conull set.) Hence $\mathrm{ch}'_B(\boldsymbol{G}) = 3$. Indeed, this argument shows that $\mathrm{ch}'_{BM}(\boldsymbol{G}) = \mathrm{ch}'_\mu(\boldsymbol{G}) = 3$. □

Further, if we let $\Delta_d = (\mathbb{Z}/2\mathbb{Z})^{*d} = \langle a_1, \ldots, a_d |\ a_1^2 = \cdots = a_d^2 = 1\rangle$, and $S = \{a_1, \ldots, a_d\}$, then $\chi'_B(\boldsymbol{G}_\infty(\Delta_d, S)) = d$ by coloring each edge according to its corresponding generator. However we have:

Theorem 8.5 ([Ma1])**.** $\mathrm{ch}'_B(\boldsymbol{G}_\infty(\Delta_d, S)) = 2d - 1$.

The values of $\mathrm{ch}'_\mu(\boldsymbol{G}_\infty(\Delta_d, S))$ for the usual product measure on the shift action and $\mathrm{ch}'_{BM}(\boldsymbol{G}_\infty(\Delta_d, S))$ are unknown for $d \geq 3$. Indeed, the following is open, and it may be that the μ-measurable and Baire measurable generalizations of the list-coloring conjecture are true aside from the exceptional case of graphs with $\Delta(\boldsymbol{G}) = 2$ as in Proposition 8.4.

Problem 8.6. *Suppose $\boldsymbol{G} = (X, G)$ is a bounded degree Borel graph where $\Delta(\boldsymbol{G}) \geq 3$. Is it true that $\mathrm{ch}'_{BM}(\boldsymbol{G}) = \chi'_{BM}(\boldsymbol{G})$? If μ is a Borel probability measure on X, is it true that $\mathrm{ch}'_\mu(\boldsymbol{G}) = \chi'_\mu(\boldsymbol{G})$?*

The special case of the list-coloring conjecture for the complete bipartite graph on n vertices $\boldsymbol{K}_{n,n}$ was known as the Dinitz Conjecture, and was proved by Galvin [Ga]. We do not know whether $\mathrm{ch}'_B(\boldsymbol{G}) = \chi'_B(\boldsymbol{G})$ for Borel bipartite graphs of degree ≥ 3.

8.2 Total coloring

A *total coloring* of a graph $\boldsymbol{G}$ is a function assigning colors to both the edges and vertices of $\boldsymbol{G}$ so that pairs consisting of two adjacent vertices, two adjacent edges, or an incident edge and vertex are all assigned distinct colors. Hence, a total coloring of a graph of maximum degree $d \geq 2$ corresponds to a vertex coloring of the obvious associated graph of maximum degree $2d$. By applying Theorem 6.5 to this graph (which obviously cannot contain connected components that are Gallai trees), we see that every Borel graph of degree $\leq d$ has a total Borel coloring with $2d$ colors. On the other hand, an easy modification of the proof of Theorem 6.12 shows that for every $d \geq 2$ there is a d-regular acyclic Borel graph with no total Borel $(2d-1)$-coloring. See the remarks after [Ma1, Theorem 3.11].

Behzad [Be] has conjectured that every bounded degree graph $\boldsymbol{G}$ has a total coloring with either $\Delta(\boldsymbol{G})+1$ or $\Delta(\boldsymbol{G})+2$ colors, and Molloy and Reed [MR] have shown that there exists a constant C such that every such graph $\boldsymbol{G}$ has a total coloring with $\Delta(\boldsymbol{G})+C$ colors. It is open whether the Baire measurable or μ-measurable generalization of Behzad's conjecture is true or even whether there is such a generalization of Molloy and Reed's result.

8.3 Unfriendly and (k, α)-colorings

An *unfriendly partition* or *unfriendly coloring* of a graph $\boldsymbol{G} = (X, G)$ is a partition of X into two sets A, B such that $\forall x \in A(|G(x) \cap B| \geq |G(x) \cap A|)$ and $\forall x \in B(|G(x) \cap A| \geq |G(x) \cap B|)$. That is, each vertex has at least as many neighbors in the opposite half of the partition as in its own half of the partition. It is easy to see that every finite graph has an unfriendly partition. Cowan and Emerson [CE] conjectured every graph has an unfriendly partition, which was shown to be false by Milner and Shelah [MS]. However, the problem remains open for countable graphs, and Weiss has asked the following more general question:

Problem 8.7 (Weiss)**.** *Does every locally countable Borel graph have a Borel unfriendly partition?*

Indeed, this question is open also for bounded degree Borel graphs. Recently the following was proved:

Theorem 8.8 ([CT])**.** *Any bounded degree Borel graph of subexponential growth admits a Borel unfriendly coloring.*

Moreover the following result in the measurable context was obtained:

Theorem 8.9 ([CT])**.** *Let $\boldsymbol{G} = (X, G)$ be a locally finite Borel graph on a standard Borel space X and let μ be a Borel probability measure on X. Assume that G is μ-measure preserving of finite cost (i.e., $\int |G(x)| \mathrm{d}\mu(x) < \infty$). Then there is a $E_{\boldsymbol{G}}$-invariant Borel set $A \subseteq X$ with $\mu(A) = 1$ such that $\boldsymbol{G}|A$ admits a Borel unfriendly coloring.*

Finally we have the following result for acyclic graphs.

Theorem 8.10 ([CMU])**.** *Let $\boldsymbol{G} = (X, G)$ be a locally finite acyclic Borel graph on a standard Borel space X. Then $\boldsymbol{G}$ has a Baire measurable unfriendly coloring and a μ-measurable unfriendly coloring, for any Borel probability measure μ on X such that $\boldsymbol{G}$ is μ-hyperfinite.*

Supposing now that $\boldsymbol{G} = (X, G)$ is a locally finite Borel graph, Conley and Kechris have suggested studying the notion of a (k, α)-*coloring* of $\boldsymbol{G}$, where $c \colon X \to k$ is a (k, α)-coloring provided that for all $x \in X$ if $c(x) = n$, then $|\{y \in G(x) : c(y) = n\}| \leq \alpha |G(x)|$. Thus, a $(k, 0)$-coloring of a graph is just the usual notion of a k-coloring, while a $(2, 1/2)$-coloring of a graph is an unfriendly coloring. Conley has proved the following theorem.

Theorem 8.11 ([Co])**.** *Suppose that (Γ, S) is an infinite marked group and $n \geq 2$. Then any free measure preserving Borel action a of Γ on a standard measure space (X, μ) is weakly equivalent to a free measure preserving Borel action b of Γ on (X, μ) such that the graph $\boldsymbol{G}(S, b)$ has a Borel $(n, 1/n)$-coloring on an invariant under b Borel set of μ-measure 1.*

Conley also uses this theorem to show the existence of invariant, random $(n, 1/n)$-colorings of $\mathbf{Cay}(\Gamma, S)$.

8.4 Graph homomorphisms and colorings

One way of regarding an n-coloring of a graph $\boldsymbol{G}$ is as a homomorphism from $\boldsymbol{G}$ to $\boldsymbol{K}_n$, the complete graph on n vertices. Thus, another way then of generalizing graph colorings is to study homomorphisms from $\boldsymbol{G}$ to other finite graphs. Indeed, several different generalized coloring notions such as fractional colorings, circular colorings, etc., can be recast in this framework (see [HN, Chapter 6]).

In recent unpublished work, Gao, Jackson, Krohne and Seward characterize the finite (or even countable) graphs $\boldsymbol{H}$ such that $\boldsymbol{G}(S, 2^{\mathbb{Z}^n}) \preceq_B H$; see the paragraph following Theorem 6.28. They also study what finite graphs $\boldsymbol{H}$ have that property that there exists a continuous homomorphism from the graph $\boldsymbol{G}(S, 2^{\mathbb{Z}^n})$ to $\boldsymbol{H}$. Their investigation has yielded several partial results, though a complete characterization is still unknown.

We end here with the following open problem which would generalize Theorem 6.3:

Problem 8.12. *For each $d \geq 1$, characterize the finite graphs $\boldsymbol{H}$ such that every acyclic Borel graph of degree $\leq d$ has a Borel homomorphism to $\boldsymbol{H}$.*

8.5 Coloring numbers

Let $\boldsymbol{G} = (X, G)$ be a graph. An *orientation* of $\boldsymbol{G}$ is an oriented graph $\boldsymbol{D}$ such that $\boldsymbol{G}_{\boldsymbol{D}} = \boldsymbol{G}$. The *coloring number* of $\boldsymbol{G}$, in symbols $\mu(\boldsymbol{G})$, is the smallest cardinal κ for which there is an orientation $\boldsymbol{D}$ of $\boldsymbol{G}$ such that for each $x \in X$, $od_{\boldsymbol{D}}(x) < \kappa$. See [EH1] for this notion.

Below let X be the disjoint union of $2^{<\mathbb{N}}$ (viewed as discrete) with the Cantor space $2^{\mathbb{N}}$. Let $\boldsymbol{G}_1$ be the graph on X in which we connect by an edge every $s \in 2^{<\mathbb{N}}$ with every $y \in 2^{\mathbb{N}}$ suhc that $s \subseteq y$.

Theorem 8.13 ([AZ, Theorem 6.2]). (1) *Let $\boldsymbol{G}$ be an analytic graph on a Polish space. Then exactly one of the following holds:*

(i) $\mu(\boldsymbol{G}) \leq \aleph_0$,

(ii) $\boldsymbol{G}_1 \preceq_c^{inj} \boldsymbol{G}$.

(2) *The set of closed graphs on $2^{\mathbb{N}}$ with $\mu(\boldsymbol{G}) \leq \aleph_0$ is a complete coanalytic set in $K((2^{\mathbb{N}})^2)$.*

As in Section 7, **(A)**, let us say that a graph $G = (X, G)$ is $\aleph_0$-*list-colorable* if for any function L that assigns to each $x \in X$ a countably infinite set $L(x)$, there is a coloring c of $\boldsymbol{G}$ such that for each x, $c(x) \in L(x)$. As a consequence of Theorem 8.13, one has the following:

Corollary 8.14 ([AZ, Corollary 6.4]). *For any analytic graph $\boldsymbol{G}$, the coloring number of $\boldsymbol{G}$ is countable iff $\boldsymbol{G}$ is $\aleph_0$-list-colorable.*

8.6 Fractional chromatic numbers

Let $\boldsymbol{G} = (X, G)$ be a graph with $\chi(\boldsymbol{G}) < \aleph_0$. For each positive integer b, we define the *b-fold chromatic number* of $\boldsymbol{G}$, in symbols $\chi^{(b)}(\boldsymbol{G})$, to be the smallest cardinality of a set Y such that there is a map $C\colon X \to \{A \subseteq Y : |A| = b\}$ with $xGy \implies C(x) \cap C(y) = \emptyset$. Clearly $\chi^{(1)}(\boldsymbol{G}) = \chi(\boldsymbol{G})$ and $\chi^{(b)}(\boldsymbol{G}) \leq b\chi(\boldsymbol{G})$. The sequence $(\chi^{(b)}(\boldsymbol{G}))_{b\geq 1}$ is subbaditive, so we define the *fractional chromatic number* of $\boldsymbol{G}$, in symbols $\chi^f(\boldsymbol{G})$, by

$$\chi^f(\boldsymbol{G}) = \lim_{b\to\infty} \frac{\chi^{(b)}(\boldsymbol{G})}{b} = \inf_{b\to\infty} \frac{\chi^{(b)}(\boldsymbol{G})}{b}.$$

Thus $\chi^f(\boldsymbol{G}) \leq \chi(\boldsymbol{G})$. See [SU] for the theory of fractional chromatic numbers. If now $\boldsymbol{G}$ is a graph on a Polish (or standard Borel space) and Φ is a class of functions on such spaces with $\chi_\Phi(\boldsymbol{G}) < \aleph_0$, we define $\chi_\Phi^{(b)}(\boldsymbol{G})$ and $\chi_\Phi^f(\boldsymbol{G})$ in the usual way, and, in particular, we define

$$\chi_c^f(\boldsymbol{G}), \chi_B^f(\boldsymbol{G}), \chi_{BM}^f(\boldsymbol{G}), \chi_\mu^f(\boldsymbol{G})$$

as before.

The fractional chromatic numbers in the descriptive context have been studied in [Mee], where the following results were proved. Note that the above quantities satisfy the following conditions:

(a) $\chi \leq \chi_\mu, \chi_{BM} \leq \chi_B$, and similarly for their χ^f counterparts.
(b) $\chi^f \leq \chi, \chi_\Phi^f \leq \chi_\Phi$, for any $\Phi \in \{B, BM, \mu\}$.
(c) $\chi, \chi_B, \chi_{BM}, \chi_\mu \in \mathbb{N}$.
(d) $\chi^f = 2 \implies \chi = 2, \chi_\Phi^f = 2 \implies \chi_\Phi = 2$, for any $\Phi \in \{B, BM, \mu\}$.

Theorem 8.15 ([Mee, Theorem 4.5.1])**.** *For each of the values of the quantities $\chi, \chi_B, \chi_{BM}, \chi_\mu$ and $\chi^f, \chi_B^f, \chi_{BM}^f, \chi_\mu^f$ in $[2, \infty)$ satisfying (a)-(d) above, there is a Borel graph $\boldsymbol{G}$ on a Polish space X and a Borel probability measure μ on X which realize all these values. If $\chi = 2$, we can choose this graph $\boldsymbol{G}$ to be acyclic.*

Note here that for any graph $\boldsymbol{G}$, with at least one edge, $\chi^f(\boldsymbol{G}) = 2 \iff \chi(\boldsymbol{G}) = 2 \iff \boldsymbol{G}$ has no odd cycles. However we have:

Theorem 8.16 ([Mee, Theorem 4.6.1])**.** *There is a Borel graph $\boldsymbol{G}$ such that $\chi_B^f(\boldsymbol{G}) = \chi_{BM}^f(\boldsymbol{G}) = 2$ but $\chi_B(\boldsymbol{G}) = \chi_{BM}(\boldsymbol{G}) = 3$.*

It is unknown if for each $n \geq 4$ there is a Borel graph $\boldsymbol{G}$ such that $\chi_B^f(\boldsymbol{G}) = 2$ but $\chi_B(\boldsymbol{G}) = n$. The following question is also raised in [Mee]:

Problem 8.17 ([Mee, Question 4.6.3]). *Calculate $\chi_B^f(\boldsymbol{G}_\infty(\mathbb{F}_n, S))$, where S is the usual symmetric set of generators for $\mathbb{F}_n$. Is it always equal ro 2?*

9. Connections with graph limits

The study of bounded degree measure preserving graphs, which were defined in the paragraph following Problem 6.13, has important connections with the theory of limits of bounded degree finite graphs, see Lovász [Lov, Part 4].

The basic link between graph limits of bounded degree finite graphs and measure preserving graphs can be described as follows. Fix below an integer $d \geq 2$ and consider graphs of bounded degree $\leq d$. For each $r \geq 1$, denote by $\mathcal{B}^{d,r}$ the set of isomorphism classes of connected, finite *rooted graphs* $\boldsymbol{B}_0 = (\boldsymbol{B}, x_0) = (X, B, x_0)$, where $\boldsymbol{B} = (X, B)$ is a finite graph on X and $x_0 \in X$ is a distinguished vertex, called the *root*, such that $\Delta(\boldsymbol{B}) \leq d$ and $d_{\boldsymbol{B}}(x, x_0) \leq r, \forall x \in X$. An isomorphism between rooted graphs $(\boldsymbol{B}, x_0)$ and $(\boldsymbol{C}, y_0)$ is an isomorphism between $\boldsymbol{B}, \boldsymbol{C}$ that sends x_0 to y_0. For each graph $\boldsymbol{G} = (X, G)$ with $\Delta(\boldsymbol{G}) \leq d$ and $x \in X$, denote by $\boldsymbol{B}_r^{\boldsymbol{G}}(x)$ the induced subgraph of $\boldsymbol{G}$ on the ball of radius r around x (in the distance $d_{\boldsymbol{G}}$). Then put for each $\boldsymbol{B}_0 \in \mathcal{B}^{d,r}$ and finite graph $\boldsymbol{G}$ with $\Delta(\boldsymbol{G}) \leq d$:

$$p_{\boldsymbol{B}_0}(\boldsymbol{G}) = \frac{|\{x \in X \colon (\boldsymbol{B}_r^{\boldsymbol{G}}(x), x) \cong \boldsymbol{B}_0\}|}{|X|}.$$

A sequence of finite connected graphs $(\boldsymbol{G}_n)$ with $\Delta(\boldsymbol{G}) \leq d$ and $|X_n| \to \infty$ is said to be *Benjamini-Schramm convergent* or *locally convergent* if for each $\boldsymbol{B}_0 \in \mathcal{B}^{d,r}$, the sequence $p_{\boldsymbol{B}_0}(\boldsymbol{G}_n)$ converges.

If $\boldsymbol{G} = (X, G)$ is now a μ-preserving Borel graph with $\Delta(\boldsymbol{G}) \leq d$, we also let for each $\boldsymbol{B}_0 \in \mathcal{B}^{d,r}$,

$$p_{\boldsymbol{B}_0}(\boldsymbol{G}) = \mu(\{x \in X \colon (\boldsymbol{B}_r^{\boldsymbol{G}}(x), x) \cong \boldsymbol{B}_0\}).$$

We then say that a locally convergent sequence of finite graphs $(\boldsymbol{G}_n)$ *locally converges* to $\boldsymbol{G}$ if for every $\boldsymbol{B}_0 \in \mathcal{B}^{d,r}$.

$$p_{\boldsymbol{B}_0}(\boldsymbol{G}_n) \to p_{\boldsymbol{B}_0}(\boldsymbol{G}).$$

It can be shown that for each locally convergent sequence of finite graphs $(\boldsymbol{G}_n)$, there is a $\boldsymbol{G}$ as above to which $(\boldsymbol{G}_n)$ locally converges; see [Lov, 19.1.2]. Such a limit $\boldsymbol{G}$ is not uniquely determined up to measure preserving isomorphism but only up to a week notion of equivalence called *local equivalence*; see [Lov, 18.5].

One way to construct such a local limit $\boldsymbol{G}$ is via the so-called *Bernoulli graphing*, see [Lov, 18.3.4]. Another is via an ultraproduct construction followed by a factoring process to obtain a measure preserving graph on a standard measure space.

When the measure preserving graph $\boldsymbol{G}$ is the local limit of a sequence $(\boldsymbol{G}_n)$, especially in the case where $\boldsymbol{G}$ is the Bernoulli graphing, one can use information on the combinatorial parameters of $\boldsymbol{G}$, in the Borel or measurable sense, e.g., independent sets, colorings, matchings, etc., to derive related information for the sequence of finite graphs $(\boldsymbol{G}_n)$. This is the basis of the so-called *Borel oracle method* of Elek-Lippner [EL]. See also Lyons [L, §4] for a related method referred as *emulation*. In the opposite direction, information about the sequence $(\boldsymbol{G}_n)$ can be sometimes transferred to a local limit $\boldsymbol{G}$. For example, it is known that for a sequence of sufficiently random $2n$-regular graphs $(\boldsymbol{G}_m)$, the independence ratio of the $\boldsymbol{G}_m$ converges to a number approximately equal to $\frac{\log 2n}{n}$, when n is large enough (see Bayati-Gamarnik-Tetali [BGT] and Frieze-Łuczak [FL]). Using ultraproducts, as in the discussion following Theorem 6.50, one can see that this implies that there is $\boldsymbol{a} \in \mathrm{FR}(\mathbb{F}_n, X, \mu)$ such that (for the usual set of generators S of $\mathbb{F}_n$) $i_\mu(\boldsymbol{G}(S, \boldsymbol{a})) \approx \frac{\log 2n}{n}$, for all large enough n.

It would take us too far afield to discuss in detail the theory of bounded degree graph limits (including other notions of convergence such the local-global convergence of Hatami-Lovász-Szegedy [HLS]) and its connections with measure preserving graphs. We only provide a (partial) list of further references for the reader who would like to pursue further this very interesting and fast growing area: [ACFK], [AH], [ATV], [AL], [AR1], [AR2], [AR3], [BGT], [BeS], [BLS], [C], [CS], [DSS], [E1] – [E7], [EL], [EL1], [FL], [GS], [HLS], [Kai], [Lov], [L], [RV], [S], [Ti2].

Part II

MATCHINGS

10. Preliminaries on matchings

A *matching* in a graph $\boldsymbol{G} = (X, G)$ is a subset M of the edges, which we view here as unordered pairs, such that no two edges in M have a common vertex. Equivalently a matching is an independent set in the line graph $\boldsymbol{L}(\boldsymbol{G})$. We denote by X_M the set of *matched vertices*, i.e., those that belong to an edge in M. If $X_M = X$, we call M a *perfect matching*. Note that if M is a matching, then we can define a fixed-point free involution $T_M \colon X_M \to X_M$ by $T_M(x) = y \iff \{x, y\} \in M$.

Given a Borel probability measure μ on a standard Borel space X, we can define the *μ-matching number* of a graph $\boldsymbol{G} = (X, G)$, by

$$m_\mu(\boldsymbol{G}) = \frac{1}{2}\sup\{\mu(X_M) \colon M \text{ is a Borel matching of } \boldsymbol{G}\}.$$

This concept is a measure-theoretic analog of the concept of *matching number* of a finite graph $\boldsymbol{G}$, which is defined as the maximum number of edges in a matching of the graph. A Borel matching M is said to be a *perfect matching μ-almost everywhere*, if X_M is $E_{\boldsymbol{G}}$-invariant and $\mu(X_M) = 1$. Thus if there is a Borel perfect matching μ-a.e., then $m_\mu(\boldsymbol{G}) = \frac{1}{2}$ and the sup is attained.

11. König's Theorem fails in the Borel context

A classical theorem of König in graph theory asserts that if $\boldsymbol{G}$ is a bipartite graph which is d-regular, for some $d \geq 2$, then $\boldsymbol{G}$ admits a perfect matching. (This is a special case of Hall's Theorem.) In A. Miller [M] the question was raised of whether this admits a Borel version, i.e., whether for each $d \geq 2$, every Borel d-regular graph with $\chi_B = 2$ admits a Borel perfect matching. A counterexample was found in Laczkovich [La88] for $d = 2$ using an ergodic theory argument. This was extended to all even d in [CK, Section 6] but the problem remained open for odd d until the recent work of Marks [Ma1], who used game theoretic methods to prove the following:

Theorem 11.1 ([Ma1, 1.5])**.** *For every $d \geq 2$, there is a d-regular, acyclic Borel graph $\boldsymbol{G} = (X, G)$ with $\chi_B(\boldsymbol{G}) = 2$ (i.e., $\boldsymbol{G}$ is bipartite in the Borel sense) on a standard Borel space X, which has no Borel perfect matching.*

The proof is an application of Theorem 6.34. With the notation of that theorem, notice that there is no Borel set $A \subseteq F([0,1]^{\Gamma * \Delta})$ such that A meets every Δ-orbit and the complement of A meets every Γ-orbit. Consider now the following Borel graph $\boldsymbol{G}$ suggested in [CK, Section 6], which for $d = 2$ was used to give a different proof of Laczkovich's result: Take $\Gamma = \Delta = \mathbb{Z}/d\mathbb{Z}$. The vertices of $\boldsymbol{G}$ are the Γ-orbits and the Δ-orbits in $F([0,1]^{\Gamma * \Delta})$. The edges of $\boldsymbol{G}$ consist of all $\{p, q\}$ such that p is a Γ-orbit, q is Δ-orbit and $p \cap q \neq \emptyset$. It is easily seen that if M is a Borel perfect matching and $A = \{x \in F([0,1]^{\Gamma * \Delta}) \colon \exists \{p, q\} \in M(\{x\} = p \cap q)\}$, then A meets every Δ-orbit and its complement meets every Γ-orbit, a contradiction.

Concerning graphs of the form $\boldsymbol{G}(S, a)$ for an infinite marked group (Γ, S) and a free Borel action a, the result in [ST] implies that if $\boldsymbol{G}(S, 2^\Gamma)$

has a Borel perfect matching, so does every $\boldsymbol{G}(S, a)$. It is easy to see that for $\Gamma = \mathbb{Z}, S = \{\pm 1)$, $\boldsymbol{G}(S, 2^\Gamma)$ does not admit a Borel perfect matching. It is also immediate that if S has an element of even order, then $\boldsymbol{G}(S, 2^\Gamma)$ has a Borel perfect matching. Recently the following was shown for the group $\mathbb{Z}^n$, $n \geq 2$, and the usual set of generators S, consisting of the n-tuples with all coordinates 0 except one which takes the value ± 1.

Theorem 11.2 (Gao-Jackson-Krohne-Seward)**.** *For $\Gamma = \mathbb{Z}^n, n \geq 2$, and S as above, the graph $G(S, 2^\Gamma)$ admits a Borel perfect matching.*

On the other hand, Marks [Ma1] shows the following:

Theorem 11.3 ([Ma1])**.** *For every $n \leq 1$, the graph $\boldsymbol{G}_\infty(\mathbb{F}_n, S)$, with $S = \{\gamma_1^{\pm 1}, \dots, \gamma_n^{\pm 1}\}$, where $\{\gamma_1, \dots, \gamma_n\}$ is a set of free generators, has no Borel perfect matching.*

12. Perfect matchings generically

Given a graph $\boldsymbol{G} = (X, G)$ on a Polish space X and a matching M of $\boldsymbol{G}$, we say that M is a *perfect matching generically* if X_M is a comeager $E_{\boldsymbol{G}}$-invariant Borel set. In contrast with Theorem 11.1 we have the following:

Theorem 12.1 ([CM3]). *Let $\boldsymbol{G} = (X, G)$ be a Borel graph on a Polish space X which is acyclic, locally finite with $\deg_{\boldsymbol{G}}(x) \geq 3, \forall x \in X$. Then there is a Borel perfect matching generically.*

Some condition on the degree of the vertices is needed as the graph $\boldsymbol{G}(S, 2^{\mathbb{Z}})$, for $S = \{\pm 1\}$, has no Borel perfect matching generically. On the other hand, in [CM3] it is shown that Theorem 12.1 is still valid under the weaker assumption that all vertices have degree at least 2 and there is no infinite injective $\boldsymbol{G}$-ray (i.e., an injective sequence (x_n) with $x_n G x_{n+1}$) such that for all *even* n we have $\deg_{\boldsymbol{G}}(x_n) = 2$. Also Theorem 12.1 fails if local finiteness is replaced by local countability, as shown in [CM3].

More recently, Marks and Unger have shown the following, which generalizes Theorem 12.1 to bipartite Borel graphs satisfying a strengthening of Hall's condition. Marks and Unger have applied this theorem to obtain some results on Baire measurable paradoxical decompositions. We discuss these results in Section 16.

Theorem 12.2 ([MU, Theorem 1.3]). *Let $\boldsymbol{G} = (X, G)$ be a locally finite bipartite Borel graph with a (not necessarily Borel) bipartition $\{B_0, B_1\}$. Suppose there exists an $\epsilon > 0$ such that for every finite set $F \subseteq B_0$ or $F \subseteq B_1$, $|N_{\boldsymbol{G}}(F)| \geq (1 + \epsilon)|F|$. Then $\boldsymbol{G}$ has a Borel perfect matching generically.*

The example $\boldsymbol{G}(S, 2^{\mathbb{Z}})$ shows that this theorem cannot be improved to have $\epsilon = 0$.

Suppose $\boldsymbol{G} = (X, G)$ is an acyclic Borel graph where every vertex has degree at least 2, and there are no infinite injective rays in $\boldsymbol{G}$ of vertices of degree equal to 2. Now along any finite path where each vertex has degree 2, a perfect matching must alternate between edges which are in and out of the matching. Thus, we can contract away such paths in an obvious way to obtain an acyclic Borel graph $\boldsymbol{G}'$ so that a generic perfect matching of $\boldsymbol{G}'$ yields a generic perfect matching of $\boldsymbol{G}$, and so no two vertices of degree 2 are adjacent in $\boldsymbol{G}'$. An easy calculation shows that Theorem 12.2 applies to $\boldsymbol{G}'$ and so we can conclude that $\boldsymbol{G}'$ has a perfect matching generically and hence so does $\boldsymbol{G}$. In particular, Theorem 12.2 implies the stronger version of Theorem 12.1 we have discussed in the paragraph following Theorem 12.1.

13. Perfect matchings almost everywhere

Recall that if $\boldsymbol{G} = (X, G)$ is a locally countable Borel graph on a standard Borel space X and μ a Borel probability measure on X, then the graph $\boldsymbol{G}$ is μ-measure preserving if for some (equivalently any) sequence of Borel involutions (T_n) with $\boldsymbol{G} = \boldsymbol{G}_{(T_n)}$, each T_n is μ-measure preserving.

A μ-measure preserving locally countable Borel graph $\boldsymbol{G} = (X, G)$ is *strictly expanding* if there is $c > 1$ such that for any Borel independent set $A \subseteq X$, if we let $N_{\boldsymbol{G}}(A) = \{x \colon \exists y \in A(xGy)\}$, then $\mu(N_{\boldsymbol{G}}(A)) \geq c\mu(A)$. We now have the following result of Lyons-Nazarov [LN], which can be viewed as a measure theoretic analog of Hall's Theorem.

Theorem 13.1 ([LN, 2.6])**.** *Let $\boldsymbol{G} = (X, G)$ be a locally finite, μ-measure preserving, bipartite, strictly expanding Borel graph on a standard measure space (X, μ). Then $\boldsymbol{G}$ admits a Borel perfect matching μ-almost everywhere.*

Note that the strictly increasing condition cannot be replaced by the condition $\mu(N_{\boldsymbol{G}}(A)) \geq \mu(A)$ as the example of the graph $\boldsymbol{G}(S, 2^{\mathbb{Z}})$, for $S = \{\pm 1\}$, shows.

Among other things, the proof of Theorem 13.1 uses a result of Elek-Lippner [EL] concerning Borel matchings with no small augmenting paths, which is interesting in its own right.

Suppose $\boldsymbol{G} = (X, G)$ is a graph and M a matching in $\boldsymbol{G}$. An *augmenting path* in $\boldsymbol{G}$ (for M) is a path $x_0, \dots, x_{2n+1}$ such that $(x_{2i}, x_{2i+1}) \notin M$, for $i \leq n$, $(x_{2i+1}, x_{2i+2}) \in M$, for $i < n$, and $x_0, x_{2n+1} \notin X_M$. A classical result in finite graph theory, due to Berge, states that if $\boldsymbol{G}$ is a finite graph, then M is a matching of $\boldsymbol{G}$ of maximum size iff there are no augmenting paths for M. We can now state the result of Elek-Lippner:

Theorem 13.2 ([EL, 1.1]). *Let $\boldsymbol{G} = (X, G)$ be a locally finite Borel graph on a standard Borel space X and let $T \geq 1$. Then for any Borel matching M of $\boldsymbol{G}$, there is a Borel matching M' of $\boldsymbol{G}$, which has no augmenting paths of length $\leq 2T + 1$ and $X_M \subseteq X_{M'}$.*

Remark 13.3. In [EL, 1.1] this result is stated for bounded degree graphs but the proof can be modified to also work for locally finite graphs.

On the other hand, using the example of Laczkovich [La88], mentioned in Section 10, Elek and Lippner show in [EL, 1.2] that there exists a 2-regular Borel graph $\boldsymbol{G}$ in which every Borel matching has an augmenting path.

Remark 13.4. In [LN, p. 1116] it is pointed out that as a consequence of Theorem 13.1, the graph discussed after Theorem 11.1 for $n = 3$, which has no Borel perfect matching, has a Borel perfect matching almost everywhere, in the following sense. Consider the Lebesgue measure λ on $[0, 1]$ and let μ be the product measure on $[0, 1]^{\Gamma * \Delta}$, where $\Gamma = \Delta = \mathbb{Z}/3\mathbb{Z}$. Then there is a Borel set A with $\mu(A) = 1$ which is invariant under the shift action of $\Gamma * \Delta$ on $[0, 1]^{\Gamma * \Delta}$, such that the graph $\boldsymbol{G}$ restricted to the Γ, Δ-orbits contained in A admits a Borel perfect matching.

Conley and B. Miller have shown the following result:

Theorem 13.5 ([CM3]). *Let $\boldsymbol{G} = (X, G)$ be an acyclic, locally countable Borel graph on a standard measure space (X, μ). If $\boldsymbol{G}$ is μ-hyperfinite and every point in an $E_{\boldsymbol{G}}$-invariant Borel sect of measure 1 has degree at least 3, then $\boldsymbol{G}$ admits a Borel perfect matching μ-almost everywhere.*

Finally, we mention the following open problem:

Problem 13.6. *Let $\boldsymbol{G} = (X, G)$ be a Borel 3-regular graph on a standard Borel space X and let μ be a Borel probability measure on X. Is there a Borel perfect matching almost everywhere?*

14. Matchings in measure preserving group actions

We use here the concepts and notation of Section 5,**(F)**. For $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ we note again that the quantity $m_\mu(\boldsymbol{G}(S, \boldsymbol{a}))$ is well-defined. We now have the following analog of Theorem 6.44 and Theorem 6.53.

Theorem 14.1 ([CKT-D, 6.1, 6.2])**.** *Let (Γ, S) be an infinite marked group. Then for any $\boldsymbol{a}, \boldsymbol{b} \in \mathrm{FR}(\Gamma, X, \mu)$ we have*

$$\boldsymbol{a} \prec \boldsymbol{b} \implies m_\mu(\boldsymbol{G}(S, \boldsymbol{a})) \leq m_\mu(\boldsymbol{G}(S, \boldsymbol{b})).$$

Moreover, for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, there is $\boldsymbol{b} \in \mathrm{FR}(\Gamma, X, \mu)$, such that $\boldsymbol{a} \sim \boldsymbol{b}$, so that $m_\mu(\boldsymbol{G}(S, \boldsymbol{a})) = m_\mu(\boldsymbol{G}(S, \boldsymbol{b}))$, and moreover the supremum in $m_\mu(\boldsymbol{G}(S, \boldsymbol{b}))$ is attained.

Concerning the value of $m_\mu(\boldsymbol{G}(S, \boldsymbol{a}))$ for $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, we have the following fact:

Proposition 14.2 ([CKT-D, 8.5])**.** *Let (Γ, S) be an amenable, infinite marked group. Then $m_\mu(\boldsymbol{G}(S, \boldsymbol{a})) = \frac{1}{2}$, for every $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$.*

The example of the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$ shows that the sup in this result might not be attained.

Lyons and Nazarov [LN] showed that if the marked group (Γ, S) is not amenable and $\mathbf{Cay}(\Gamma, S)$ is bipartite, then the graph $\boldsymbol{G}_\infty(\Gamma, S)$ admits a perfect matching almost everywhere, with respect to the usual product measure on $[0, 1]^\Gamma$.

Finally Csóka-Lippner [CL] eliminated the bipartite assumption.

Theorem 14.3 ([CL])**.** *Let (Γ, S) be a non-amenable, marked group. Then the graph $\boldsymbol{G}_\infty(\Gamma, S)$ admits a Borel perfect matching almost everywhere (with respect to the usual product measure).*

The combination of Theorems 14.1, 14.2 and 14.3, now shows the following:

Corollary 14.4. *Let (Γ, S) be an infinite marked group. Then for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, $m_\mu(\boldsymbol{G}(S, \boldsymbol{a})) = \frac{1}{2}$ and there is $\boldsymbol{b} \in \mathrm{FR}(\Gamma, X, \mu)$ such that $\boldsymbol{a} \sim \boldsymbol{b}$ and $\boldsymbol{G}(S, \boldsymbol{b})$ admits a Borel perfect matching μ-almost everywhere.*

The next question is of course to find out for which infinite marked groups (Γ, S) and $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, the graph $\boldsymbol{G}(S, \boldsymbol{a})$ admits a Borel perfect matching μ-almost everywhere. The answer is trivially positive for *any* $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, if S has an element of even order. On the other hand it is easy to see that for $\Gamma = \mathbb{Z}$, $S = \{\pm 1\}$ and any *weakly mixing* action $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, there is no Borel perfect matching almost everywhere. We have seen in Theorem 11.2 (and the paragraph preceding it) that for $\Gamma = \mathbb{Z}^n$, $n \geq 2$, and the usual set of generators S, the graphs $\boldsymbol{G}(S, \boldsymbol{a})$ actually admit a Borel perfect matching. Timar [Ti1] had actually proved earlier that the graph $\boldsymbol{G}_\infty(\mathbb{Z}^n, S)$, $n \geq 2$, admits a Borel perfect matching μ-almost everywhere, where μ is the product measure. This leads to the following open problem raised in [LN]:

Problem 14.5. *For which infinite, amenable marked groups (Γ, S) does the graph $\boldsymbol{G}_\infty(\Gamma, S)$ admit a Borel perfect matching almost everywhere (with respect to the usual product measure)?*

Finally, we mention that in [CKT-D, 8.6] it is shown that if $\Gamma = \langle a, b | a^3 = b^3 = 1\rangle = (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ and $S = \{a, b, a^{-1}, b^{-1}\}$, then for any $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$, the graph $\boldsymbol{G}(S, \boldsymbol{a})$ admits a Borel perfect matching μ-almost everywhere.

Remark 14.6. For a graph $\boldsymbol{G} = (X, G)$ a *vertex cover* is a subset $V \subseteq X$ such that every edge is incident to some vertex in V. If (X, μ) is a standard measure space, we define the *μ-vertex covering number* of $\boldsymbol{G}$ by

$$v_\mu(\boldsymbol{G}) = \inf\{\mu(V) \colon V \text{ is a Borel vertex cover}\}.$$

Since V is a vertex cover iff $X \setminus V$ is an independent set, it follows that $v_\mu(\boldsymbol{G}) = 1 - i_\mu(\boldsymbol{G})$.

When $\boldsymbol{G}$ is finite, bipartite and μ is the counting measure on X, then a result of König (which implies the matching theorem mentioned in the beginning of Section 10), asserts that $v_\mu(\boldsymbol{G}) = m_\mu(\boldsymbol{G})$.

Consider now the case where $\boldsymbol{G} = \boldsymbol{G}(S, \boldsymbol{a})$. Call the action $\boldsymbol{a}$ a *König action* if it satisfies $v_\mu(\boldsymbol{G}) = m_\mu(\boldsymbol{G})$. It is easy to see that: (i) $i_\mu(\boldsymbol{G}) \leq m_\mu(\boldsymbol{G})$ (if A is a Borel independent set and $s \in S$, then the set of all $\{a, s \cdot a\}$, for $s \in S$ and $a \in A$ forms a matching); (ii) $m_\mu(\boldsymbol{G}) \leq v_\mu(\boldsymbol{G})$ (since if M is a Borel matching and V is a Borel vertex cover, then there is a Borel vertex cover $V' \subseteq V$ such that V' meets every edge of M in exactly one point). Thus since $m_\mu(\boldsymbol{G}), i_\mu(\boldsymbol{G}) \leq \frac{1}{2}$, clearly $\boldsymbol{G}$ is König iff $i_\mu(\boldsymbol{G}) = \frac{1}{2}$. Thus if $\mathbf{Cay}(\Gamma, S)$ is not bipartite, $\boldsymbol{G}$ is not König (see Proposition 6.48). If $\mathbf{Cay}(\Gamma, S)$ is bipartite and Γ is amenable, then $\boldsymbol{G}$ is König (see Theorem 6.51 and paragraph following it). Finally if $\mathbf{Cay}(\Gamma, S)$ is bipartite and Γ is not amenable, then for $\boldsymbol{a} = \boldsymbol{a}_{\Gamma,0}$, $\boldsymbol{G}$ is not König (by Theorem 6.52 and Theorem 6.44), while if $\boldsymbol{a} = \boldsymbol{a}_{\Gamma,\infty}$, then $\boldsymbol{G}$ is König (by Theorem 6.49 and Theorem 6.44).

15. Invariant random perfect matchings

As in Section 5,**(G)**, the preceding results have applications to problems in probability theory concerning invariant random perfect matchings in Cayley graphs.

To start with, we state the following result of Abért, Csóka, Lippner and Terpai, see [CL, 1.2]:

Theorem 15.1 ([CL, 1.2])**.** *For every infinite marked group* (Γ, S)*, the graph* $\mathbf{Cay}(\Gamma, S)$ *admits a perfect matching.*

Actually, as it is shown in [CL], this holds for any infinite, bounded degree, connected, vertex transitive graph.

Remark 15.2. It is not hard to show that for any infinite marked group (Γ, S), if there is $\boldsymbol{a} \in \mathrm{FR}(\Gamma, X, \mu)$ with $m_\mu(\boldsymbol{G}(S, \boldsymbol{a})) = \frac{1}{2}$, then $\mathbf{Cay}(\Gamma, S)$ admits a perfect matching. To see this, note that by a simple compactness argument it is enough to show that any finite subset $F \subseteq \Gamma$ is contained in X_M, for some matching M of $\mathbf{Cay}(\Gamma, S)$. Now let M_n be a Borel matching of $\boldsymbol{G}(S, \boldsymbol{a})$ such that if $X_n = X_{M_n}$, then $\mu(X_n) > 1 - 2^{-n}$. Then if $Y_m = \bigcap_{n \geq m} X_n$, we have that $Y_0 \subseteq Y_1 \subseteq Y_2 \dots$ and $\mu(Y_m) \to 1$. Put $Y = \bigcup_m Y_m$, so that $\mu(Y) = 1$, and thus there is an invariant under $\boldsymbol{a}$ Borel set $Z \subseteq Y$ with $\mu(Z) = 1$. If $x \in Z$, then for some m and all $\gamma \in F$, we have that $\boldsymbol{a}(\gamma, x) \in Y_m$, so, using the freeness of $\boldsymbol{a}$, we conclude that there is some M as above. In particular, this shows that Corollary 14.4 implies Theorem 15.1.

We next consider the space $2^{E(\Gamma, S)}$, where $E(\Gamma, S)$ is the set of all edges of $\mathbf{Cay}(\Gamma, S)$ (considered as unordered pairs). We view this as the space of all subsets of $E(\Gamma, S)$. We denote by $M(\Gamma, S)$ the nonempty, closed subspace of $2^{E(\Gamma,S)}$ consisting of all perfect matchings. As usual Γ acts by shift on $M(\Gamma, S)$ and so does $\mathrm{Aut}_{\Gamma,S}$. We can therefore define as before the concept of a Γ (resp., $\mathrm{Aut}_{\Gamma,S}$)-*invariant, random perfect matching* $\mathbf{Cay}(\Gamma, S)$ as a Borel probability measure on $M(\Gamma, S)$ which is invariant under the shift action. As in Section 5,**(G)**, there is an $\mathrm{Aut}_{\Gamma,S}$-invariant, random perfect matching iff there is a Γ-invariant, random perfect matching. Finally, we define as in Section 5,**(G)** the concept of invariant, random perfect matching of $\mathbf{Cay}(\Gamma, S)$ which is a factor of IID.

The connection of perfect matchings almost everywhere for the graph $\boldsymbol{G}(S, \boldsymbol{a})$ and Γ-invariant, random perfect matchings of $\mathbf{Cay}(\Gamma, S)$ that are factors of $\boldsymbol{a}$ carries over from the case of colorings to that of matchings, mutatis mutandis. Thus we have the following corollary of Theorems 14.3 and 14.4:

Corollary 15.3. *Let (Γ, S) be an infinite marked group. Then $\mathbf{Cay}(\Gamma, S)$ admits a Γ-invariant, random perfect matching.*

Moreover, if Γ is not amenable, then $\mathbf{Cay}(\Gamma, S)$ admits a Γ-invariant, random perfect matching which is a factor of IID.

It is clear from the discussion following Corollary 14.4 that for $\Gamma = \mathbb{Z}$, $S = \{\pm 1\}$, $\mathbf{Cay}(\Gamma, S)$ does not admit a Γ-invariant, random perfect matching which is a factor of IID, while the Cayley graph of $\mathbb{Z}^n$ does, when $n \geq 2$. It is unknown for which infinite amenable (Γ, S) there is a Γ-invariant, random perfect matching of $\mathbf{Cay}(\Gamma, S)$ which is a factor of IID.

16. Paradoxical decompositions and matchings

Suppose $a\colon \Gamma \times X \to X$ is an action of a group Γ on a set X. Then two sets $A, B \subseteq X$ are said to be *a-equidecomposable* if there is a partition of A into finitely many sets $\{A_1, \ldots, A_n\}$ and group elements $\alpha_1, \ldots, \alpha_n \in \Gamma$ so that the sets $\alpha_i \cdot A_i$ are disjoint, and $B = \alpha_1 \cdot A_1 \cup \ldots \cup \alpha_n \cdot A_n$. It is easy to see that equidecomposability is an equivalence relation. The action a is said to have a *paradoxical decomposition* if X can be partitioned into two sets $\{A, B\}$ so that A and B are each a-equidecomposable with X.

Let $a\colon \Gamma \times X \to X$ be a Borel action of a countable group Γ on a standard Borel space X. Given a finite symmetric set $S \subseteq \Gamma$ and two Borel sets $A, B \subseteq X$, we can form the Borel graph

$$\boldsymbol{G}(S, a, A, B) = (\{0\} \times A \cup \{1\} \times B, G(S, a, A, B))$$

where

$$(i, x) G(S, a, A, B)(j, y) \iff i \neq j \wedge \exists \gamma \in S(\gamma \cdot x = y).$$

It is easy to see that A and B are a-equidecomposable using group elements from S if and only if $\boldsymbol{G}(S, a, A, B)$ has a perfect matching.

Similarly, if a is as above, to each finite set $S \subseteq \Gamma$ we can form a Borel graph $\boldsymbol{G}_p(S, a)$ which has a perfect matching if and only if a has a paradoxical decomposition using group elements from S. Let a' be the action of $\mathbb{Z}/3\mathbb{Z} \times \Gamma$ on three copies $\{0, 1, 2\} \times X$ of X via $a'((n, \gamma), (i, x)) = (n + i \pmod 3, a(\gamma, x))$. Then a has a paradoxical decomposition if and only if $\{0\} \times X$ is a'-equidecomposable with $\{1, 2\} \times X$. Hence, we define

$$\boldsymbol{G}_p(S, a) = \boldsymbol{G}(\{1, 2\} \times S, a', \{0\} \times X, \{1, 2\} \times X).$$

The relationship we have described above between paradoxical decompositions and perfect matchings gives a standard way of proving Tarski's theorem that the left translation action of a nonamenable countable group has a paradoxical decomposition. If Γ has no Følner sequence, there is a finite symmetric set $S \subseteq \Gamma$ such that for every finite set $F \subseteq \Gamma$, $|SF| \geq 2|F|$ (see [CC, Theorem 4.9.2]). Hence, the graph $\boldsymbol{G}_p(S, a)$ satisfies the requirement of Hall's matching theorem.

We can also use this connection between matchings and paradoxical decompositions to give interesting examples of Borel graphs that have no Borel perfect matchings. Suppose (X, μ) is a standard measure space, Γ is a countable group, and S is a finite symmetric subset of Γ. Let $a\colon \Gamma \times X \to X$ be a Borel μ-measure preserving action of Γ on X. Then there can be no paradoxical decomposition of X into μ-measurable sets since this would imply $\mu(X)$ is both 1 and 2. Hence, for every $S \subseteq \Gamma$, there can be no Borel perfect matching of the graph $\boldsymbol{G}_p(S, a)$ restricted to any μ-conull set.

Indeed, by considering the graphs defined above associated to free measure preserving actions of free groups $\mathbb{F}_n$ of increasing rank we have the following strong refutation of Hall's matching theorem in the measurable context, which is an interesting contrast to Theorem 13.1:

Proposition 16.1. *For every $n \geq 1$, there is a μ-measure preserving bounded degree Borel graph $\boldsymbol{G} = (X, G)$ on a standard measure space (X, μ) with $\chi_B(\boldsymbol{G}) = 2$ such that for every finite set $F \subseteq Y$, the set of neighbors $N_{\boldsymbol{G}}(F)$ of elements of F satisfies $|N_{\boldsymbol{G}}(F)| \geq n|F|$, but $\boldsymbol{G}$ has no Borel perfect matching μ-almost everywhere.*

In contrast to the fact that a measure preserving action has no measurable paradoxical decompositions, we have the following result of Grabowski, Máthé and Pikhurko for equidecomposability of bounded sets of the same measure in $\mathbb{R}^n$ for $n \geq 3$. This theorem is proved by applying Lyons and Nazarov's Theorem 13.1 to appropriate graphs of the form $\boldsymbol{G}(S, a, A, B)$:

Theorem 16.2 ([GMP1], [GMP3])**.** *Let $n \geq 3$ and suppose $A, B \subseteq \mathbb{R}^n$ are bounded Lebesgue measurable sets of the same measure with nonempty interiors. Then A and B are equidecomposable by rigid motions using Lebesgue measurable pieces.*

More recently, in every dimension $n \geq 1$, Grabowski, Máthé and Pikhurko have proved the following:

Theorem 16.3 ([GMP2]). *Suppose $n \geq 1$, $A, B \subseteq \mathbb{R}^n$ are bounded, non-null, Lebesgue measurable sets with the same measure, and the upper Minkowski dimension of the boundaries of A and B is less than n. Then A and B are equidecomposable by translations using Lebesgue measurable pieces.*

This theorem builds on work of Laczkovich [La92] who proved its analogue for equidecompositions constructed using the Axiom of Choice. As an application, Theorem 16.3 can be used to show that the Tarski Circle Squaring Problem can be solved using Lebesgue measurable pieces.

Addendum. Marks and Unger have now obtained in [MU1] a constructive solution of the Tarski Circle Squaring Problem (that avoids the use of the Axiom of Choice) by showing that Theorem 16.3 holds with the pieces of the equidecomposition being Borel.

In the context of Baire category, Dougherty and Foreman have shown that if $n \geq 2$ and $a\colon \mathbb{F}_n \times X \to X$ is a free action of a free group of rank n on a Polish space X by homeomorphisms, then a has a paradoxical decomposition with pieces having the property of Baire [DF]. Dougherty-Foreman originally used this result to show that the Banach-Tarski paradox can be performed using pieces with the Baire property, solving a problem of Marczewki from the 1930s. More recently, Marks and Unger have used Theorem 12.2 to generalize Dougherty and Foreman's result to all Borel actions with paradoxical decompositions, as follows:

Theorem 16.4 ([MU, Theorem 1.1]). *Suppose $a\colon \Gamma \times X \to X$ is an action of group Γ on a Polish space X, and for each $\gamma \in \Gamma$, the function $x \mapsto \gamma \cdot x$ is Borel. Then if a has a paradoxical decomposition, then a has a paradoxical decomposition using pieces with the Baire property.*

Marks and Unger have also used this result to give a Baire category solution to the dynamical von Neumann-Day problem [MU, Theorem 1.2].

Wehrung [Weh] has shown that there is no paradoxical decomposition of the action of the group of rotations on the unit sphere in $\mathbb{R}^3$ using 4 pieces which have the Baire property. However, using the axiom of choice one can prove there are paradoxical decompositions of the sphere (without the Baire property) using 4 pieces. Thus, Wehrung's result can be applied to show that there are Borel graphs of the form $\boldsymbol{G}_p(S, a)$ that have a perfect matching, but no perfect matching generically. This also shows that a certain amplification of pieces needed in the proof of Marks and Unger's theorem is necessary.

To conclude this section, we provide another proof of Dougherty and Foreman's theorem on the existence of Baire measurable paradoxical decompositions of actions of $\mathbb{F}_n$. Our argument is inspired by the proof of Theorem 12.1 of Conley and Miller that acyclic locally finite Borel graphs of degree ≥ 3 have Borel perfect matchings generically. The framework we use to prove the theorem is quite general and we obtain a few other corollaries.

We begin with an abstract lemma. If X and Y are Polish and $\boldsymbol{G} = (X, G)$ is a Borel graph on X, then we let $\mathcal{F}_{\boldsymbol{G}}(X, Y)$ be the standard Borel space of finite partial functions from X to Y with $\boldsymbol{G}$-connected domains. For $B \subseteq \mathcal{F}_{\boldsymbol{G}}(X, Y)$, let $\mathcal{P}_B(X, Y)$ be the set of partial functions f from X to Y such for every connected component C of $G|\operatorname{dom}(f)$, C is finite and $f|C \in B$.

Lemma 16.5. *Suppose X is Polish, $\boldsymbol{G} = (X, G)$ is a locally finite Borel graph, Y is countable, and $B \subseteq \mathcal{F}_{\boldsymbol{G}}(X, Y)$ is Borel. Suppose also that for all $g \in \mathcal{P}_B(X, Y)$ and $x \in X$, there is a extension $g' \in \mathcal{P}_B(X, Y)$ of g with $x \in \operatorname{dom}(g')$. Then there is an increasing sequence of partial Borel functions $f_0 \subseteq f_1 \subseteq \ldots \in \mathcal{P}_B(X, Y)$ such that $\operatorname{dom}(\bigcup_i f_i)$ is an $E_{\boldsymbol{G}}$-invariant Borel comeager set.*

Proof. Let $(T_n)_{n \in \mathbb{N}}$ be a set of Borel involutions whose graphs union to $E_{\boldsymbol{G}}$, and fix a countable basis $(U_m)_{m \in \mathbb{N}}$ of open subsets of X. Let g_0 be the empty function. Now, given g_i, we construct g_{i+1} as follows. Let Z_i be the set of $C \in [E_{\boldsymbol{G}}]^{<\infty}$ such that $\exists p \in B$ with $\operatorname{dom}(p) = C$, $g_i \cup p \in \mathcal{P}_B(X, Y)$, and C is a connected component of $\boldsymbol{G}|\operatorname{dom}(g_i \cup p)$. Note that Z_i is Borel and every $x \in X$ is in some set in Z_i by assumption. By Proposition 4.10, there is a Borel $\mathbb{N}$-coloring c of the intersection graph on $[E_{\boldsymbol{G}}]^{<\infty}$. Hence we can partition Z_i into countably many Borel sets $Z_{i,j} = \{C \in Z_i : c(\{x : d_{\boldsymbol{G}}(x, C) \leq 1\}) = j\}$, where if C, C' are distinct elements of $Z_{i,j}$, then no two elements of C and C' are adjacent. Let (n, m) be the ith pair of natural numbers. Then there must be some k such that $T_n(\bigcup\{C \colon C \in Z_{i,k}\})$ is nonmeager in U_m. Now since $\{(C, p) \in Z_{i,k} \times \mathcal{F}_{\mathbf{G}}(X, Y) : \operatorname{dom}(p) = C \ \& \ \forall x \in \operatorname{dom}(p) \cap \operatorname{dom}(g_i)(p(x) = g_i(x))\}$ is Borel, by applying Lusin-Novikov uniformization (see [K95, 18.10]), we can find a Borel $g_{i+1} \in \mathcal{P}_B(X, Y)$ extending g_i whose domain is $\operatorname{dom}(g_i) \cup \bigcup\{C \colon C \in Z_{i,k}\}$.

To finish, note that $T_n(\operatorname{dom}(\bigcup_i g_i))$ is comeager, for every n, since it is nonmeager in every basic open set U_m by construction. Hence, $\operatorname{dom}(\bigcup_i g_i)$ contains a Borel $\boldsymbol{G}$-invariant comeager Borel set, namely $\bigcap_n T_n(\operatorname{dom}(\bigcup_i g_i))$. Our desired functions f_i then are the restrictions of the g_i to this set. $\square$

We now use this lemma to prove the following.

Lemma 16.6. *Suppose $\boldsymbol{G} = (X, G)$ is a locally finite acyclic Borel graph, in which every vertex has degree ≥ 3. Then there exists a partial Borel function $f \subseteq G$ on X with no infinite forward orbits, such that $\mathrm{ran}(f)$ is an $E_{\boldsymbol{G}}$-invariant comeager set, and for all $y \in \mathrm{ran}(f)$, we have $|f^{-1}(y)| \geq |\deg_{\boldsymbol{G}}(y) - 2|$.*

Proof. Let $(T_n)_{n\in\mathbb{N}}$ be a countable set of Borel involutions such that $\boldsymbol{G} = \boldsymbol{G}_{(T_n)}$, and where $T_0(x) = x$ for all x. Given a partial function p from X to $\mathbb{N}$, we can associate to it the function $\hat{p}$ where if $p(x) = n$, then $\hat{p}(x) = T_n(x)$. We will use Lemma 16.5 to obtain a set of partial functions f_i from X to $\mathbb{N}$. Setting $h = \bigcup_i f_i$, our desired function f will be defined by setting $f(x) = \hat{h}(x)$, if $\hat{h}(x) \neq x$, and undefined otherwise.

We will apply Lemma 16.5 to the graph $\boldsymbol{G}' = (X, G')$ where $xG'y$ if $0 < d_{\boldsymbol{G}}(x, y) \leq 2$. We define the set $B \subseteq \mathcal{F}_{\boldsymbol{G}'}(X, \mathbb{N})$ to be the set of $p \in \mathcal{F}_{\boldsymbol{G}'}(X, Y)$ such that

(i) For every $x \in \mathrm{dom}(p)$, there is an m such that $\hat{p}^{m+1}(x) = \hat{p}^m(x)$. That is, the forward orbit of each $x \in \mathrm{dom}(p)$ under $\hat{p}$ eventually reaches some y with $\hat{p}(y) = y$.

(ii) If $x \in X$ has at least 2 $\boldsymbol{G}$-neighbors (note, not $\boldsymbol{G}'$-neighbors) in $\mathrm{dom}(p)$, then $x \in \mathrm{dom}(p)$ and $|\hat{p}^{-1}(x)| \geq |\deg_{\boldsymbol{G}}(x) - 2|$.

Of course, (i) ensures that the f we construct will have no infinite forward orbits, and (ii) is to satisfy the requirement $|f^{-1}(y)| \geq |\deg_{\boldsymbol{G}}(y) - 2|$ for $y \in \mathrm{ran}(f)$.

To apply Lemma 16.5 it remains to show that for every $g \in \mathcal{P}_B(X, \mathbb{N})$ and $z \in X$, there is an extension $g' \in \mathcal{P}_B(X, \mathbb{N})$ of g with $z \in \mathrm{dom}(g')$. To see this, set $g_0 = g$ and $A_0 = \{z\}$. Now, inductively, given A_i, let B_{i+1} be the set of points x neighboring an element of A_i that are not in $\mathrm{dom}(g_i)$. Since G is acyclic, inductively, each point $x \in B_{i+1}$ must have a unique element $y \in A_i$ as a neighbor. Define $g_{i+1} \supseteq g_i$ so that $\hat{g}_{i+1}(x) = y$ for all such x and y. Now let A_{i+1} be the set of $x \in B_{i+1}$ such that x has 2 neighbors in $\mathrm{dom}(g_i)$. It is easy to see that all elements of B_i used in this construction come from the same finite connected component of $\boldsymbol{G}'$ as x does. Hence, there must be a stage i where $A_i = \emptyset$ and setting $g' = g_i$ gives the desired function. $\square$

We can now conclude with the Dougherty-Foreman result.

Corollary 16.7 ([DF]). *Suppose $n \geq 2$, $a\colon \mathbb{F}_n \times X \to X$ is a free Borel action of $\mathbb{F}_n$ on a Polish space X. Then a has a paradoxical decomposition using sets with the property of Baire.*

Proof. Let $S = \{\gamma_1^{\pm 1}, \dots, \gamma_n^{\pm 1}\}$, where $\{\gamma_1, \dots, \gamma_n\}$ is a free set of generators for $\mathbb{F}_n$. The graph $\boldsymbol{G}(S, a) = (X, G)$ is an acyclic Borel graph everywhere of degree ≥ 4. Let f be as in Lemma 16.6 for this graph and set $A = \mathrm{ran}(f)$. It suffices to construct injective Borel functions $g, h\colon A \to A$ such that $g, h \subseteq \boldsymbol{G}(S, a)|A$ and A is the disjoint union of $\mathrm{ran}(g)$ and $\mathrm{ran}(h)$, since such functions are in 1-1 correspondence with pairs of injective functions with paradoxical decompositions of a using group elements from S.

Let $A_0 = \{x \in A : x \notin \mathrm{dom}(f)\}$. Given A_i, let $A_{i+1} = \{x \in A : f(x) \in A_i\}$. Since f has no infinite forward orbits, $\bigcup_i A_i = A$. Fix a Borel linear ordering of X. We construct g and h in countably many steps. At step i we define g and h on A_i. For each $x \in A_i$, if $x \in \mathrm{dom}(f)$, $y = f(x)$ is not yet in $\mathrm{ran}(g)$ or $\mathrm{ran}(h)$, and x is the least element in $f^{-1}(y)$, then define $g(x) = y$. Otherwise, $f^{-1}(x)$ contains at least two different elements in A_{i+1}. Define $g(x)$ to be the least element of $f^{-1}(x)$ and $h(x)$ to be the greatest element of $f^{-1}(x)$. $\square$

Moreover Dougherty and Foreman discuss in [DF] some variants of the Banach-Tarski paradox involving disjoint open subsets of the unit sphere which can be rearranged with rigid motions so that their closures have "paradoxical" properties. It is possible to also prove these results using the above technique by paying careful attention to the Borel complexity of the sets we have used above.

We now note that the technique we have used above to prove Corollary 16.7 can also be used to prove several other results. For example, a similar argument can be used to prove Theorem 12.1 as corollary of Lemma 16.6; given f as in Lemma 16.6, and A_i as in the above proof, inductively define a Borel perfect matching of $\boldsymbol{G}|A$ in countably many steps by matching $x \in A_i$ to the least element of $f^{-1}(x)$ if x is not yet in an edge in the matching. Similarly, from Lemma 16.6 one can also quickly prove the theorem that every locally finite acyclic Borel graph has a Baire measurable 3-coloring, since each element of A_{i+1} has at most 2 neighbors in $\bigcup_{i<j} A_j$. This corresponds to the acyclic case of Theorem 5.6.

Finally, we note that Lemma 16.6 also yields a proof of Theorem 6.17, by once again coloring all edges adjacent to elements of A_i at step i as above.

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