ONE-ENDED SPANNING SUBFORESTS AND TREEABILITY OF GROUPS

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Abstract. We show that several new classes of groups are measure strongly treeable. In particular, finitely generated groups admitting planar Cayley graphs, elementarily free groups, and $\mathrm{Isom}(\mathbf{H}^2)$ and all its closed subgroups. In higher dimensions, we also prove a dichotomy that the fundamental group of a closed aspherical 3-manifold is either amenable or has strong ergodic dimension 2. Our main technical tool is a method for finding measurable treeings of Borel planar graphs by constructing one-ended spanning subforests in their planar dual. Our techniques for constructing one-ended spanning subforests also give a complete classification of the locally finite p.m.p. graphs which admit Borel a.e. one-ended spanning subforests.

Contents

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0. Introduction

This article is a contribution to the study of measured and Borel equivalence relations, in terms of their graphed structures, with applications in the measured group theory of countable and locally compact groups.

Dramatic progress has been realized in the study of discrete groups in relation with topological and geometric ideas over the course of the 20th century, from the early works of Klein, Poincaré, Dehn, Nielsen, Reidemeister and Schreier for instance, to Bass-Serre theory and Thurston's Geometrization program as well as hyperbolic groups and the emergence of geometric group theory as a distinct area of mathematics under the impulse of Gromov. In his monograph [Gro93], Gromov outlined his program of understanding countable discrete groups up to quasi-isometry (e.g., cocompact lattices in the same locally compact second countable group G). In the same text Gromov also introduced the parallel notion of measure equivalence (**ME**) between countable discrete groups [Gro93, 0.5.E], the most emblematic example being lattices in G. Two groups are ME if they admit commuting, free, measure-preserving actions on a nonzero Lebesgue measure space with finite measure fundamental domains. This concept is strongly connected with orbit equivalence (**OE**) in ergodic theory ([Fur99], [Gab02b, Th. 2.3]; see [Gab05, Fur11] for surveys on ME and OE).

The history of orbit equivalence itself can be traced back to the work of Dye [Dye59, Dye63] stemming on the group-measure-space von Neumann algebra of Murray and von Neumann [MvN36]. The abstract and basic objects connecting this turn out to be the standard measure-class preserving equivalence relations, as axiomatized by Feldman-Moore [FM77]. A major milestone is the elucidation of the connections between five properties (see [Con76, CK77, OW80, CFW81]): *the following are equivalent: (1) hyperfiniteness of the group-measure-space von Neumann algebra, (2) hyperfiniteness of the equivalence relation, (3) amenability of the equivalence relation, (4) orbit equivalence with a* $\mathbf{Z}$*-action, and (5) when the action is assumed to be probability measure preserving (**p.m.p.**) and free, amenability of the acting group.* As a consequence, the measure equivalence class of $\mathbf{Z}$ consists exactly in all infinite amenable groups. Thus such a useful geometric invariant as the growth becomes apparently irrelevant in measured group theory insofar as amenability is concerned, although our Theorem 2.6 leads us to reconsider this observation.

Much of progress in orbit equivalence has been realized since the 80's following a suggestion of A. Connes at a conference in Santa Barbara in 1978 (see [Ada90]) of studying equivalence relations $\mathcal{R}$ with an additional piece of data: a measurably-varying simplicial complex structure on each equivalence class (aka a **complexing** [AG21]). The 1-dimensional complexings are known as **graphings**. Their acyclic version (**treeings**) were originally studied by S. Adams [Ada90, Ada88]. Both are constitutive of the theory of **cost** [Lev95, Gab00], since this is defined in terms of graphings and treeings allow to compute it [Gab00, Théorème 1]. Graphings and treeings have also played a crucial role in the theory of structurings on countable Borel equivalence relations [JKL02].

Amenability, seen from the perspective of orbit equivalence, can be rephrased as the capability of embellishing almost every orbit (equivalence class) with a measurably-varying oriented line structure [Dye59, OW80, CFW81]. Alternatively, it is easy to equip the classes of any hyperfinite equivalence relations with a **one-ended** tree structure. As a kind of converse, in the p.m.p. context (which will be our context in the introduction through Theorem 9) any treeing of an amenable equivalence relation is (class-wise) at most two-ended ([Ada90]).

Beyond amenability, the simplest groups from the measured theoretic point of view are the **treeable** ones: those admitting a free p.m.p. action whose orbit equivalence relation can be equipped with a treeing (for more precise definitions of the various notions of treeability, see Appendices A and B). This is an extremely rich and still mysterious class of groups (see the survey part of [Gab05]). By a theorem of Hjorth [Hjo06], this is precisely the class of groups Γ that are ME with a free group $\mathbf{F}_n$. This family splits into four ME-classes: $n = 0$ when Γ is finite, $n = 1$ when Γ is infinite amenable, and $n = 2$ or $n = \infty$ according to their cost belonging to $(1, \infty)$ or $\{\infty\}$ [Gab00].

The first substantial example of a treeable group apart from free products of amenable groups is the fundamental group $\pi_1(\Sigma)$ of a closed hyperbolic surface Σ. Indeed, both $\pi_1(\Sigma)$ and $\mathbf{F}_2$ share the property of being isomorphic to lattices in $G = \mathrm{SL}(2, \mathbf{R})$. It follows that $\pi_1(\Sigma)$ admits at least one treeable free action, namely the natural action by multiplication $\pi_1(\Sigma) \curvearrowright G/\mathbf{F}_2$ (with Haar measure). It is a longstanding question of [Gab00, Question VI.2] whether treeable groups are **strongly treeable**[1], i.e., whether **all** their free p.m.p. actions are treeable. This question has been open for twenty years, even for $\pi_1(\Sigma)$ (see [Gab02b, Question p. 176]), and we solve it in this case. This is our first main result:

Theorem 1. Surface groups are strongly treeable. More generally, finitely generated groups admitting a planar Cayley graph are strongly treeable.

A more general statement can be found in Theorem 4.4.

The introduction of the notion of "measurable free factor" in [Gab05], led to the production of some new examples of treeable groups, such as branched surface groups. These are examples from a family that we will discuss now. The **elementarily free groups** are those groups with the same first-order theory as the free

[1]Observe that unfortunately for consistency of terminology, in [Gab00] the terms "arborable" and "anti-arborable" are used instead of the current better terms "strongly treeable" and "non-treeable" respectively, that we will adopt here.

group $\mathbf{F}_2$. We shall use their description (when finitely generated) as fundamental groups of certain tower spaces (according to the results of [Sel06] and [KM98], made utterly complete in [GLS20] – see §5 for more details). A careful analysis of their virtual structure allowed [BTW07] to apply results from [Gab05] to a finite index subgroup in order to achieve their treeability. The question of their strong treeability has remained open since then; we resolve it:

Theorem 2. Finitely generated elementarily free groups are strongly treeable.

Strong treeability has a number of consequences which do not follow from treeability. In particular, if Γ is a strongly treeable group, then by [Gab00, Prop. VI.21] Γ satisfies the fixed price conjecture [Gab00, Question I.8]. The groups having a planar Cayley graph appearing in Theorem 1, such as the cocompact Fuchsian triangle groups, give the first new examples of groups of fixed price greater than 1 since [Gab00]. (The cocompact Fuchsian triangle groups admit finite index subgroups which are surfaces, but both strong treeability and fixed price are not known to pass to finite index super-groups.)

The arguments developed for the above theorem gave us as a by-product the following interesting claim (Corollary 5.11). Let $r \geq 3$ and $\Gamma_1, \Gamma_2, \cdots, \Gamma_r$ be countable groups and let $\gamma_i \in \Gamma_i$ be an infinite order element for each $i = 1, 2, \cdots, r$. If the Γ_i are all treeable or strongly treeable, then the same holds not only for their free product, but also for its quotient by the normal subgroup generated by the product of the γ_i: $(\Gamma_1 * \Gamma_2 * \cdots * \Gamma_r) / \langle\langle \prod_{i=1}^{r} \gamma_i \rangle\rangle$.

It is worth mentioning that treeability has also had an impact in the theory of von Neumann algebras. Popa's discovery [Pop06] of the first II_1 factor with trivial fundamental group (namely the group von Neumann algebra $L(\mathrm{SL}(2, \mathbf{Z}) \ltimes \mathbf{Z}^2))$ used his rigidity-deformation theory to establish uniqueness of the HT Cartan subalgebra, thereby reducing the problem to the study of the orbit equivalence relation of the treeable action of $\mathrm{SL}(2, \mathbf{Z})$ on the 2-torus. Later on, Popa and Vaes extended drastically the class of groups whose free p.m.p. actions lead to uniqueness of the Cartan subalgebra [PV14a, PV14b], and thus the study of the group-measure space von Neumann algebra of these actions boils down to that of the action up to OE. This class contains free groups, non-elementary hyperbolic groups, their directs products and all groups that are ME with these groups. The class of countable groups satisfying 2-cohomology vanishing for cocycle actions on II_1 factors is speculated to coincide with the class of treeable groups [Pop18, Remarks 4.5].

However, it is far from the case that all countable groups are treeable. The first examples of non-treeable groups are the infinite Kazhdan property (T) groups [AS90] and the non-amenable cost 1 groups [Gab00, Th. 4], and more generally all non-amenable groups with $\beta_1^{(2)} = 0$ [Gab02a, Proposition 6.10]. The random graph formulation of treeability is the existence of an invariant probability measure supported on the set of spanning trees on the group; in [PP00], Pemantle and Peres prove that a non-amenable direct product of infinite groups can have no such probably measure. The non treeability of non-amenable direct products also follows from the theory of cost [Gab00].

The study of non-treeable groups must involve higher dimensional geometric objects. Recall that the **geometric dimension** of a countable group Γ is the smallest dimension of a contractible complex on which Γ acts freely. Analogously, the second author introduced in [Gab02a, Déf. 3.18] the **geometric dimension** of a p.m.p. standard equivalence relation $\mathcal{R}$ on a probability measure space (X, μ) as the smallest dimension of a measurable bundle of contractible simplicial complexes over X on which $\mathcal{R}$ acts **smoothly**, i.e., for which there exists a Borel fundamental domain for the action of $\mathcal{R}$. The **ergodic dimension of a group** Γ [Gab02a, Déf. 6.4] is the smallest geometric dimension among all of its free p.m.p. actions (see [Gab21] for more on this notion). Being infinite treeable is thus a synonym of having ergodic dimension 1. The ergodic dimension of a group is an ME-invariant, it is bounded above by its virtual geometric dimension is non-increasing when taking subgroups.

We say that Γ has **strong ergodic dimension** d if all its free p.m.p. actions have geometric dimension d. Honesty and humility force us to admit our ignorance: not a single group is known with two free p.m.p. actions having different geometric dimensions.

By Ornstein-Weiss [OW80], all infinite amenable groups have strong ergodic dimension 1. Recall that non vanishing of ℓ^2-Betti numbers produces lower bounds for the ergodic dimension (see [Gab02a, Prop. 5.8, Cor. 3.17]). It follows for instance that (with $p_i \geq 2$) $\Gamma = \mathbf{F}_{p_1} \times \mathbf{F}_{p_2} \times \cdots \times \mathbf{F}_{p_d}$ or $\Gamma = (\mathbf{F}_{p_1} \times \mathbf{F}_{p_2} \times \cdots \times \mathbf{F}_{p_d}) * \mathbf{F}_k$ have strong ergodic dimension d [Gab02a, pp. 126-127] while $\Gamma \times \mathbf{Z}$ has strong ergodic dimension $d+1$ and $\mathrm{Out}(\mathbf{F}_n)$ has strong ergodic dimension $2n-3$ [GN21, Theorem 1.6, Theorem 1.1].

It is not hard to check that the 3-dimensional manifolds with one of the eight geometric structures of Thurston have ergodic dimension at most 2. For instance the fundamental group of a closed hyperbolic 3-dimensional manifold is a cocompact lattice in $\mathrm{SO}(3,1)$. It is ME with non-compact lattices in $\mathrm{SO}(3,1)$, which have geometric dimension 2 and zero first ℓ^2-Betti number, and thus they have ergodic dimension at most 2. We prove a strong dichotomy theorem for the ergodic dimension of fundamental groups of aspherical manifolds of dimension 3.

Theorem 3 (Theorem 6.2)**.** Suppose Γ is the fundamental group of a closed (i.e., compact without boundary) aspherical (possibly non-orientable) manifold of dimension 3. Then either

(1) Γ is amenable, or
(2) Γ has strong ergodic dimension 2.

If one removes the assumption of asphericity, the Kneser-Milnor theorem [Kne29, Mil62] decomposes the fundamental group of an orientable closed 3-dimensional manifold as a free product of amenable groups and groups to which Theorem 3 applies. It follows that

Theorem 4. If M is an orientable closed 3-dimensional manifold, then $\Gamma = \pi_1(M)$ has strong ergodic dimension $\in \{0, 1, 2\}$.

Considering the orientation covering, one deduces that the ergodic dimension is at most 2 if M^3 is non-orientable. However, strongness also holds in this case, but this shall be treated elsewhere [Gab21]. This touches the delicate open question whether strong ergodic dimension is an invariant of commensurability. If one allows M to have boundary components, then we loose a priori the strongness but we still obtain that every p.m.p. free action of $\Gamma = \pi_1(M)$ has geometric dimension at most 2.

It is worth noting that all our results are proved without appealing to the recent progress in Thurston's geometrization theorem. In higher dimensions, we also obtain a non-trivial bound :

Theorem 5 (Theorem 6.1)**.** Suppose $\Gamma = \pi_1(M)$ is the fundamental group of a compact aspherical manifold M (possibly with boundary) of dimension at least 2. Then all free p.m.p. actions of Γ have ergodic dimension at most $\dim(M) - 1$.

The theory of measure equivalence has been extended beyond countable discrete groups to include all unimodular locally compact second countable (**lcsc**) groups (see the nice survey [Fur11] and see [KKR17] for basic invariance properties). The investigation about their treeability began with a result of Hjorth [Hjo08, Theorem 0.5] stating that the products $G_1 \times G_2$ of infinite lcsc groups are non treeable unless both are amenable. He observed that amenable groups are strongly treeable and asks which other lcsc groups satisfy this property [Hjo08, p. 387]. We produce the first progress in this study since then:

Theorem 6 (See Corollary 4.2)**.** $\mathrm{Isom}(\mathbf{H}^2)$, $\mathrm{PSL}_2(\mathbf{R})$, and $\mathrm{SL}_2(\mathbf{R})$ are all strongly treeable, as are all of their closed subgroups.

While the notion of treeability extends in the natural way to orbit equivalence relations of actions $G \curvearrowright (X, \mu)$ of lcsc groups, an equivalent way of conceiving a treeing in this context is by introducing a cross section $B \subseteq X$ (see section A) to which the restriction $\mathcal{R}_{\restriction B}$ of the orbit equivalence relation $\mathcal{R}_{G \curvearrowright (X,\mu)}$ has countable classes and is treeable.

This result gives as a by-product the first examples of non trivial fixed price for connected lcsc groups (Definition A.9). In contrast, fixed price 1 for the direct product of some lcsc groups with the integers is obtained in [AM21]. Once a Haar measure is prescribed on G, the quantity $\frac{\mathrm{cost}(\mathcal{R}_{\restriction B})-1}{\mathrm{covolume}(B)}$ does not depend on the cross section B, since the restrictions are pairwise stably orbit equivalent (Proposition A.8). A remarkable consequence of Theorem 6 is that this quantity is also independent of the free p.m.p. action $G \curvearrowright (X, \mu)$, for each of these groups:

Theorem 7 (See Corollary 4.2 and Remark 4.3)**.** The groups $G = \mathrm{Isom}(\mathbf{H}^2)$, $\mathrm{PSL}_2(\mathbf{R})$, and $\mathrm{SL}_2(\mathbf{R})$, and their closed subgroups have fixed price

A central fascination in our study is, given a graphing, the hunt for a subgraphing all of whose connected components are acyclic and have exactly one end (hereafter named a **one-ended spanning subforest**, since several connected components are usually necessary for covering a single class). Besides their intrinsic interest, one-ended spanning subforests prove to be extremely useful in our applications, e.g., Theorems 3.6, 5.1, 6.1, and 6.2.

Much of the technical work in the paper consists of finding new techniques for constructing Borel a.e. one-ended spanning subforests of locally finite Borel graphs.

In particular, in the case of locally finite p.m.p. graphs, we give a complete characterization of what graphs admit Borel a.e. one-ended spanning subforests.

Theorem 8 (Theorem 2.1)**.** Suppose that $\mathcal{G}$ is a measure preserving aperiodic locally finite Borel graph on a standard probability space (X, μ). Then $\mathcal{G}$ has a Borel μ-a.e. one-ended spanning subforest $\mathcal{T} \subseteq \mathcal{G}$ iff $\mathcal{G}$ is μ-nowhere two-ended.

Here, **μ-a.e. one-ended spanning subforest** means that the set of vertices of the one-ended trees of $\mathcal{T}$ has full μ-measure in X; while **μ-nowhere two-ended** means that the set of vertices of the two-ended connected components of $\mathcal{G}$ has measure zero in X.

The search for subtrees or subforests has attracted enormous attention in another but related mathematical field: the theory of random graphs and percolation (already alluded to in our introduction to non-treeable groups). Thus, for example Pemantle [Pem91] introduced the spanning forest FUSF for $\mathbf{Z}^d$, obtained as the limiting measure of the uniform spanning tree on large finite pieces of the lattice. He proved that it is connected if and only if $d \leq 4$. The use of various subforests such as the (wired and free) minimal spanning forests (WMSF, FMSF) or the (wired and free) uniform spanning (WUSF, FUSF) forests are of crucial significance in the theory of percolation on graphs [BLPS01, LPS06]. The WMSF is an instance of a random one-ended spanning subforest. The authors of [LPS06] have shown the equivalence of WMSF$\neq$FMSF with the famous conjecture of Benjamini-Schramm [BS96] whether $p_c \neq p_u$. The mean valency of the FUSF equals two plus twice the first ℓ^2-Betti number [Lyo09, Corollary 4.12]. If we knew that adding a random graph of arbitrarily small mean valency could make the FUSF forest connected, then it would solve the cost vs first ℓ^2-Betti number question of [Gab02a, p. 129]. See also [GL09] and [Tim19] for further connections between treeability and percolation.

Theorem 8 relies on Elek and Kaimanovich's characterization of when a locally finite p.m.p. graph $\mathcal{G}$ is **μ-hyperfinite** (i.e., when there exists a μ-conull subset X_0 of X such that the connectedness equivalence relation $\mathcal{R}_\mathcal{G}$ of $\mathcal{G}$ is hyperfinite once restricted to X_0). On the contrary, we say $\mathcal{G}$ is **μ-nowhere hyperfinite** if there does not exist a positive measure subset A of X such that $(\mathcal{R}_\mathcal{G})_{\upharpoonright A}$ is hyperfinite.

We also use Theorem 8 to give an interesting dual statement to the well-known part (1) of the following theorem that a graph is μ-hyperfinite if and only if it has complete sections of arbitrarily large measure whose induced subgraphs are finite.

Theorem 9 (Theorem 1.3)**.** Let $\mathcal{G}$ be a locally finite p.m.p graph on a standard probability space (X, μ).

(1) $\mathcal{G}$ is μ-hyperfinite if and only if for every $\epsilon > 0$, there exists a Borel complete section $A \subseteq X$ for $\mathcal{R}_\mathcal{G}$ with $\mu(A) > 1 - \epsilon$ so that $\mathcal{G}_{\upharpoonright A}$ has finite connected components.
(2) $\mathcal{G}$ is μ-nowhere hyperfinite if and only if for every $\epsilon > 0$ there exists a Borel complete section $A \subseteq X$ for $\mathcal{R}_\mathcal{G}$ with $\mu(A) < \epsilon$ such that $\mathcal{G}_{\upharpoonright A}$ is $\mu_{\upharpoonright A}$-nowhere hyperfinite.

By significantly relaxing the measure preserving hypothesis, we arrived at the study of Borel graphings and their behavior with respect to various Borel probability measures that are not necessarily measure preserving. Although the measure μ is not a priori related to the Borel equivalence relation E under consideration, some properties may hold up to discarding a set of μ-measure 0. In this non p.m.p. context, we obtain one-ended spanning subforests out of Borel planar graphs (see Definition 3.5).

Theorem 10 (See Corollary 3.7)**.** Let $\mathcal{G}$ be a locally finite Borel graph on X whose connected components are planar. Let μ be a Borel probability measure on X. If $\mathcal{G}$ is μ-nowhere two-ended, then $\mathcal{G}$ has a Borel a.e. one-ended spanning subforest.

Without the "μ-nowhere two-ended" assumption, one still gets the existence of a spanning subforest on a μ-conull set (see Theorem 3.6).

We say that a Borel equivalence relation E on X is **measure treeable** if for each Borel probability measure μ on X, there a μ-conull subset X_0 of X such that the restriction of E to X_0 is Borel treeable (Definition A.1). Observe that when the classes of E are countable then each such μ is dominated by a quasi-invariant measure $\mu' = \sum_{i=1}^{\infty} 2^{-i} g_{i*}\mu$, where $\{g_i\}_{i=1}^{\infty}$ is an enumeration of some countable group G that generates E (see [FM77]). Note that μ and μ' have the same E-invariant null sets. See also Proposition A.4.

A lcsc group G is called **measure strongly treeable** if all orbit equivalence relations generated by all free Borel actions of G are measure treeable (Definition B.1).

In this context, using Theorem 10, our Theorems 1, 2 and 6 take indeed a much stronger non p.m.p. form (see Corollary 4.2, Corollary 4.4, and Theorem 5.1):

Theorem 11. The following groups are measure strongly treeable
(1) $\mathrm{Isom}(\mathbf{H}^2)$, $\mathrm{PSL}_2(\mathbf{R})$, and $\mathrm{SL}_2(\mathbf{R})$, and all of their closed subgroups.
(2) Finitely generated groups admitting a planar Cayley graph.
(3) Finitely generated elementarily free groups.

Our proof follows an idea from [BLPS01] (also used in [Gab05]) of finding an a.e. one-ended spanning subforest in the planar dual (see §3). Suppose $\mathcal{G}$ is a locally finite graph admitting an accumulation-point free planar embedding into $\mathbf{R}^2$. Then it is easy to see that subtreeings of $\mathcal{G}$ are in one-to-one correspondence with one-ended subforests in the planar dual of $\mathcal{G}$. We use this correspondence to show the measure treeability of the above groups G by converting the problem of treeing an action of G into finding an a.e. one-ended spanning subforest of the planar dual of graphings of the action which are Borel planar.

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(-*2*-) circulated. [2]

[2]**Comment:** We started this research some years ago ???
Thank some institutions/conferences where we discussed that ? Warwick ? Salon des vins bio de Condrieu (?)

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1. Elek's refinement of Kaimanovich's Theorem for measured graphs

If $\mathcal{G}$ is a locally finite Borel graph on a standard probability space (X, μ), then the **(vertex) isoperimetric constant of** $\mathcal{G}$ is the infimum of $\mu(\partial_{\mathcal{G}} A)/\mu(A)$ over all Borel subsets $A \subseteq X$ of positive measure such that $\mathcal{G}_{\restriction A}$ has finite connected components. Here, $\partial_{\mathcal{G}} A$ denotes the set of vertices in $X \setminus A$ which are $\mathcal{G}$-adjacent to a vertex in A. If the measure μ is $\mathcal{R}_{\mathcal{G}}$-quasi-invariant then the isoperimetric constant of $\mathcal{G}$ can be equivalently phrased in terms of the associated Radon-Nikodym cocycle (see [KM04, §8]).

In [Kai97], Kaimanovich established the equivalence between μ-hyperfiniteness of a measured equivalence relation $\mathcal{R}$ and vanishing of the isoperimetric constant of all bounded graph structures on $\mathcal{R}$. In [Ele12], Elek sharpened Kaimanovich's Theorem by establishing the following characterization of hyperfiniteness for a fixed measured graph $\mathcal{G}$.

Theorem 1.1 (Elek [Ele12])**.** *Let $\mathcal{G}$ be a locally finite Borel graph on a standard probability space (X, μ). Then $\mathcal{G}$ is μ-hyperfinite if and only if for every positive measure Borel subset $X_0 \subseteq X$, the isoperimetric constant of $\mathcal{G}_{\restriction X_0}$ is* 0.

While the theorem in [Ele12] is stated for measure preserving bounded degree graphs, it can easily be extended to all locally finite graphs which are not necessarily measure preserving. For the convenience of the reader we indicate the proof.

Proof of Theorem 1.1. Suppose first that $\mathcal{G}$ is μ-hyperfinite. Let $X_0 \subseteq X$ be a Borel set of positive measure and let $\mathcal{H} = \mathcal{G}_{\restriction X_0}$. Then $\mathcal{H}$ is μ-hyperfinite, so after ignoring a null set we can find finite Borel subequivalence relations $\mathcal{R}_0 \subseteq \mathcal{R}_1 \subseteq \cdots$ with $\mathcal{R}_{\mathcal{H}} = \bigcup_n \mathcal{R}_n$. Since $\mathcal{H}$ is locally finite, given $\epsilon > 0$, we may find n large enough so that $\mu(A_n) > \mu(X_0)(1-\epsilon)$, where $A_n = \{x \in X_0 : \mathcal{H}_x \subseteq [x]_{\mathcal{R}_n}\}$ and $\mathcal{H}_x$ denotes the set of $\mathcal{H}$-neighbors of x. Then $\mathcal{H}_{\restriction A_n} \subseteq \mathcal{R}_n$, so $\mathcal{H}_{\restriction A_n}$ has finite connected components. In addition, $\mu(\partial_{\mathcal{H}} A_n)/\mu(A_n) < \epsilon/(1-\epsilon)$, so as $\epsilon > 0$ was arbitrary this shows the isoperimetric constant of $\mathcal{H}$ is 0.

Assume now that for every positive measure Borel subset $X_0 \subseteq X$ the isoperimetric constant of $\mathcal{G}_{\restriction X_0}$ is 0. To show that $\mathcal{G}$ is μ-hyperfinite it suffices to show that for any $\epsilon > 0$ there exists a Borel set $Y \subseteq X$ with $\mu(Y) \geq 1 - \epsilon$ such that $\mathcal{G}_{\restriction Y}$ has finite connected components (since then we can find a sequence of such sets Y_n, $n \in \mathbf{N}$, with $\mu(Y_n) \geq 1 - 2^{-n}$, so by Borel-Cantelli $\mathcal{R}_{\mathcal{G}} = \liminf_n \mathcal{R}_{\mathcal{G}_{\restriction Y_n}}$ is μ-hyperfinite). Given $\epsilon > 0$, by Zorn's Lemma we can find a maximal collection $\mathcal{A}$ of pairwise disjoint nonnull Borel subsets of X subject to

(i) $\mathcal{G}_{\restriction \bigcup \mathcal{A}}$ has finite connected components;
(ii) $\mu(\partial_{\mathcal{G}}(\bigcup \mathcal{A})) \leq \epsilon\mu(\bigcup \mathcal{A})$;
(iii) If $A, B \in \mathcal{A}$ are distinct, then no vertex in A is adjacent to a vertex in B.

Let $Y = \bigcup \mathcal{A}$. We now claim that the set $X_0 = X \setminus (Y \cup \partial_{\mathcal{G}} Y)$ is null. Otherwise, by hypothesis we may find a Borel set $A_0 \subseteq X_0$ of positive measure such that $\mathcal{G}_{\restriction A_0}$ has finite connected components and $\mu(\partial_{\mathcal{G}_{\restriction X_0}} A_0) < \epsilon\mu(A_0)$. But then the collection $\mathcal{A}_0 = \mathcal{A} \cup \{A_0\}$ satisfies (i)-(iii) in place of $\mathcal{A}$, contradicting maximality of $\mathcal{A}$. Thus, $\mu(Y) = 1 - \mu(\partial_{\mathcal{G}} Y) \geq 1 - \epsilon\mu(Y) \geq 1 - \epsilon$, and $\mathcal{G}_{\restriction Y}$ has finite connected components, which finishes the proof. □

Remark 1.2. We have used the vertex isoperimetric constant, whereas [Ele12] uses the edge isoperimetric constant. The relationship is as follows. Let $\mathcal{G}$ be a graph on (X, μ) and let M_r be the Borel σ-finite measures on $\mathcal{G}$ given by $M_r(D) = \int_X |D^x|\, d\mu$. The **edge isoperimetric constant** of $\mathcal{G}$ is the infimum of $M_r(\partial^e_{\mathcal{G}} A)/\mu(A)$ over all Borel subsets $A \subseteq X$ of positive measure such that $\mathcal{G}_{\restriction A}$ has finite connected components. Here, $\partial^e_{\mathcal{G}} A$ is the set of all edges of $\mathcal{G}$ having one endpoint in A and one in $X \setminus A$. (Note that since $\partial^e_{\mathcal{G}} A$ is symmetric, one obtains the same definition if in place of M_r one uses the measure $M_l(D) = \int_X |D_x|\, d\mu$.) It is then easy to see that if μ is $\mathcal{G}$-quasi-invariant, and if $\mathcal{G}$ is bounded (meaning that $\mathcal{G}$ is bounded degree and the Radon-Nikodym cocycle $\rho : \mathcal{R}_{\mathcal{G}} \to \mathbf{R}^+$ associated to μ is essentially bounded on $\mathcal{G}$) then for any positive measure Borel subset $X_0 \subseteq X$, the edge isoperimetric constant of $\mathcal{G}_{\restriction X_0}$ vanishes if and only if the vertex isoperimetric constant of $\mathcal{G}_{\restriction X_0}$ vanishes.

The combinatorial core of the proof of the forward direction of Theorem 1.1 is the fact that a graph $\mathcal{G}$ is μ-hyperfinite if and only if there are arbitrarily large sets on which its restriction is finite. There is a dual statement for μ-nowhere hyperfiniteness.

Theorem 1.3. *Let $\mathcal{G}$ be a p.m.p. locally finite Borel graph on a standard probability space (X, μ).*

(1) $\mathcal{G}$ is μ-hyperfinite if and only if for every $\epsilon > 0$, there exists a Borel complete section $A \subseteq X$ for $\mathcal{R}_{\mathcal{G}}$ with $\mu(A) > 1 - \epsilon$ so that $\mathcal{G}_{\restriction A}$ has finite connected components.

(2) $\mathcal{G}$ is μ-nowhere hyperfinite if and only if for every $\epsilon > 0$ there exists a Borel complete section $A \subseteq X$ for $\mathcal{R}_{\mathcal{G}}$ with $\mu(A) < \epsilon$ such that $\mathcal{G}_{\restriction A}$ is $\mu_{\restriction A}$-nowhere hyperfinite.

Proof. The new content of the theorem is the forward direction of (2). (In our proof of Theorem 1.1 we indicated how to prove part (1)). A key point in our proof is the use of Corollary 2.11 below on the existence of maximal hyperfinite one-ended spanning subforests, which we establish later.

By Corollary 2.11 (whose assumption is satisfied when $\mathcal{G}$ is μ-nowhere hyperfinite) we may find a Borel μ-a.e. one-ended spanning subforest $\mathcal{T} \subseteq \mathcal{G}$ such that $\mathcal{R}_{\mathcal{T}}$ is μ-maximal among the $\mathcal{G}$-connected, μ-hyperfinite equivalence subrelations of $\mathcal{R}_{\mathcal{G}}$.

Let $A_1 = X$ and for $n \geq 1$ let $A_{n+1} = \{x \in A_n : \deg_{\mathcal{T}_{\restriction A_n}}(x) \geq 2\}$. Then $A_1 \supseteq A_2 \supseteq \cdots$ and $\mu(\bigcap_n A_n) = 0$ since $\mathcal{T}$ is one-ended, so after ignoring a null set we may assume that $\bigcap_n A_n = \varnothing$. Observe that $\mathcal{R}_{\mathcal{T}_{\restriction A_n}} = (\mathcal{R}_{\mathcal{T}})_{\restriction A_n}$ for all n. Given $\epsilon > 0$, let n be so large that $\mu(A_n) < \epsilon/2$. Since $\mathcal{G}$ is μ-nowhere hyperfinite and $\mathcal{T}$ is μ-hyperfinite, the set $\mathcal{G} \setminus \mathcal{R}_{\mathcal{T}}$ meets almost every connected component of $\mathcal{T}$.

We may therefore find a subset $\mathcal{G}_0 \subseteq \mathcal{G} \setminus \mathcal{R}_{\mathcal{T}}$ which is incident with almost every connected component of $\mathcal{T}$ such that the set B, of vertices incident with $\mathcal{G}_0$, has measure $\mu(B) < \epsilon/2n$. (Finding $\mathcal{G}_0$ is easy when $\mathcal{T}$ is ergodic; in general we can simply use the ergodic decomposition of $\mathcal{T}$.)

Claim. *$\mathcal{T} \cup (\mathcal{G}_{\restriction B})$ is μ-nowhere hyperfinite.*

Proof of the claim. Suppose otherwise. Then we may find a non-null $\mathcal{R}_{\mathcal{T}\cup(\mathcal{G}_{\restriction B})}$-invariant set D such that $(\mathcal{R}_{\mathcal{T}\cup(\mathcal{G}_{\restriction B})})_{\restriction D}$ is hyperfinite. Then the equivalence relation $\mathcal{Q} := (\mathcal{R}_{\mathcal{T}\cup\mathcal{G}_{\restriction B}})_{\restriction D} \sqcup (\mathcal{R}_{\mathcal{T}})_{\restriction(X\setminus D)}$ is $\mathcal{G}$-connected and μ-hyperfinite. By our choice of B, each component of $\mathcal{T} \cup (\mathcal{G}_{\restriction B})$ contains more than one component of $\mathcal{T}$, so $\mathcal{Q}$ properly contains $\mathcal{R}_{\mathcal{T}}$ since D is non-null. This contradicts the maximality of $\mathcal{R}_{\mathcal{T}}$. □

For each $x \in B$ let $\pi(x) \in A_n$ denote the unique vertex in A_n which is closest to x with respect to the graph metric in $\mathcal{T}$, and let p_x denote the unique shortest path through $\mathcal{T}$ from x to $\pi(x)$. The length of each p_x is at most $n-1$, so if we let C denote the set of all vertices which lie along p_x for some $x \in B$, then $\mu(C) \le n\mu(B) < \epsilon/2$. Let $A = A_n \cup C$. Then $\mu(A) < \epsilon$ and

$$\mathcal{R}_{\mathcal{G}_{\restriction A}} \supseteq \mathcal{R}_{\mathcal{T}_{\restriction A}} \vee \mathcal{R}_{\mathcal{G}_{\restriction B}} = (\mathcal{R}_{\mathcal{T}})_{\restriction A} \vee \mathcal{R}_{\mathcal{G}_{\restriction B}} \supseteq (\mathcal{R}_{\mathcal{T}\cup(\mathcal{G}_{\restriction B})})_{\restriction A},$$

so that $\mathcal{G}_{\restriction A}$ is $\mu_{\restriction A}$-nowhere hyperfinite since $\mathcal{T} \cup (\mathcal{G}_{\restriction B})$ is μ-nowhere hyperfinite. □

2. One-ended spanning subforests

In this section, we characterize exactly when a locally finite probability measure preserving Borel graph has a one-ended spanning subforest.

Theorem 2.1 (For p.m.p. graphings)**.** *Suppose that $\mathcal{G}$ is a measure preserving aperiodic locally finite Borel graph on a standard probability space (X, μ). Then $\mathcal{G}$ has a Borel a.e. one-ended spanning subforest iff $\mathcal{G}$ is μ-nowhere two-ended.*

We further conjecture the following strengthening of this theorem for graphs which are not necessarily measure preserving. In this more general setting, the correct generalization of (μ-a.e.) aperiodicity is μ-nowhere smoothness of $\mathcal{G}$; we say that $\mathcal{G}$ is **μ-nowhere smooth** if there is no positive measure Borel subset of X which meets each $\mathcal{G}$-component in at most one point.

Conjecture 2.2 (Non necessarily p.m.p. graphings)**.** *Suppose that $\mathcal{G}$ is a μ-nowhere smooth locally finite Borel graph on a standard probability space (X, μ). Then $\mathcal{G}$ has a Borel a.e. one-ended spanning subforest iff $\mathcal{G}$ is μ-nowhere two-ended.*

We know that the forward direction of the above conjecture is true by Lemma 2.4 below. The reverse direction is known to be true in the case when $\mathcal{G}$ is hyperfinite by Lemma 2.10, and when $\mathcal{G}$ is acyclic by the following theorem of [CMTD16].

Theorem 2.3 (Non necessarily p.m.p. treeings [CMTD16, Theorem 1.5])**.** *Suppose that $\mathcal{G}$ is an acyclic, aperiodic locally finite Borel graph on a standard probability space (X, μ). If $\mathcal{G}$ is μ-nowhere two-ended, then $\mathcal{G}$ has a Borel a.e. one-ended spanning subforest.*

We will begin by proving the forward direction of Theorem 2.1 (and also Conjecture 2.2). An easy argument shows that the graph associated to a free measure preserving action of $\mathbf{Z}$ cannot have a Borel a.e. one-ended spanning subforest; such a subforest must come from removing a single edge from each connected component of the graph. This set of edges would witness the fact that the graph $\mathcal{G}$ is smooth, contradicting our assumption that the action of $\mathbf{Z}$ is measure-preserving. Our argument is a simple generalization of this idea.

Lemma 2.4. *Suppose that $\mathcal{G}$ is a μ-nowhere smooth locally finite Borel graph on a standard probability space (X, μ). If there is a set of positive measure on which $\mathcal{G}$ is two-ended, then $\mathcal{G}$ does not admit a Borel a.e. one-ended spanning subforest.*

Proof. By restricting to and renormalizing a Borel $\mathcal{G}$-invariant subset of positive measure, we may assume that $\mathcal{G}$ is everywhere two-ended and has a Borel a.e. one-ended spanning subforest $\mathcal{T}$. We will now show $\mathcal{G}$ is smooth. Let Y be the set of connected $C \in [\mathcal{R}_\mathcal{G}]^{<\mathbf{N}}$ such that removing C from $\mathcal{G}$ disconnects its connected component into exactly two infinite pieces. Recall that $[\mathcal{R}_\mathcal{G}]^{<\mathbf{N}}$ is the Borel set of finite subsets of X made of $\mathcal{R}_\mathcal{G}$-equivalent points. By taking a countable coloring of the intersection graph on Y (see [KM04, Lemma 7.3] and [CM16, Proposition 2]), we may find a Borel set $Z \subseteq Y$ which meets every connected component of $\mathcal{G}$ and so that distinct $C, D \in Z$ are pairwise disjoint and if C and D are in the same connected component, then $|C| = |D|$. By discarding a smooth set, we may assume Z meets each connected component of $\mathcal{G}$ infinitely many times. Let $\mathcal{H}$ be the graph on Z where $C \mathrel{\mathcal{H}} C'$ if C and C' are in the same connected component of $\mathcal{G}$ and there is no $D \in Z$ such that removing D from $\mathcal{G}$ places C and C' in different connected components. Note that $\mathcal{H}$ is 2-regular.

Now let $Z' \subseteq Z$ be the set of $C \in Z$ such that there exists a $\mathcal{H}$-neighbor D of C and a component F of $\mathcal{T}$, such that C meets F, but D does not meet F.

It is easy to see that Z' meets each connected component of $\mathcal{G}$ and is finite (else $\mathcal{T}$ is not a one-ended spanning subforest), but then $\mathcal{G}$ is smooth. ☐

Our proof of the reverse direction of Theorem 2.1 splits into two cases based on Theorem 1.1. In particular, it will suffice to prove Theorem 2.1 for μ-hyperfinite graphs, and graphs having positive isoperimetric constant.

2.1. **Measure preserving graphs with superquadratic growth.** We begin with a lemma giving a sufficient condition for a graph to possess a one-ended spanning subforest. (In fact, this condition can be shown to be equivalent to the existence of such a subforest)

Let f be a partial function from a set X into itself, and let $y \in X$. The **back-orbit** of y under f is the set of all $x \in \mathrm{dom}(f)$ for which there is some $n \geq 0$ with $f^n(x) = y$.

Lemma 2.5. *Suppose that $\mathcal{G}$ is a locally finite Borel graph on a standard probability space (X, μ), and there are partial Borel functions $f_0, f_1, \ldots \subseteq \mathcal{G}$ such that*

(1) $\sum_i \mu(\mathrm{dom}(f_i)) < \infty$
(2) $\bigcup \mathrm{dom}(f_i) = X$

(3) Every f_i is aperiodic and has finite back-orbits.

(4) For every i and $x \in \mathrm{dom}(f_i)$ there is a $j \geq i$ such that $f_i(x) \in \mathrm{dom}(f_j)$.

Then $\mathcal{G}$ has a Borel a.e. one-ended spanning subforest.

Proof. As usual, we may assume μ is $\mathcal{G}$-quasi-invariant. By (1), (2) and a Borel-Cantelli argument there is a $\mathcal{G}$-invariant conull Borel set A such that for every $x \in A$ there are only finitely many i such that $x \in \mathrm{dom}(f_i)$. Define $n(x)$ for $x \in A$ to be the largest i such that $x \in \mathrm{dom}(f_i)$. Now define $f \colon A \to A$ by $f(x) = f_{n(x)}(x)$. We claim that f generates a one-ended spanning subforest of $\mathcal{G}_{\restriction A}$. To see this, note f is aperiodic since each f_i is aperiodic by (3), and by (4) the value of $n(x)$ is non-decreasing along orbits of f. We also see that f has finite back-orbits by induction since the f_i do. □

Now we apply this lemma to show that any measure preserving graph of superquadratic growth has a Borel a.e. one-ended spanning subforest.

Theorem 2.6. *Suppose that $\mathcal{G}$ is a measure preserving locally finite Borel graph on a standard probability space (X, μ) of superquadratic growth, so there is a $c > 0$ such that for every $x \in X$, $|B_r(x)| \geq cr^2$ where $B_r(x)$ is the ball of radius r around x in $\mathcal{G}$. Then $\mathcal{G}$ has a Borel a.e. one-ended spanning subforest.*

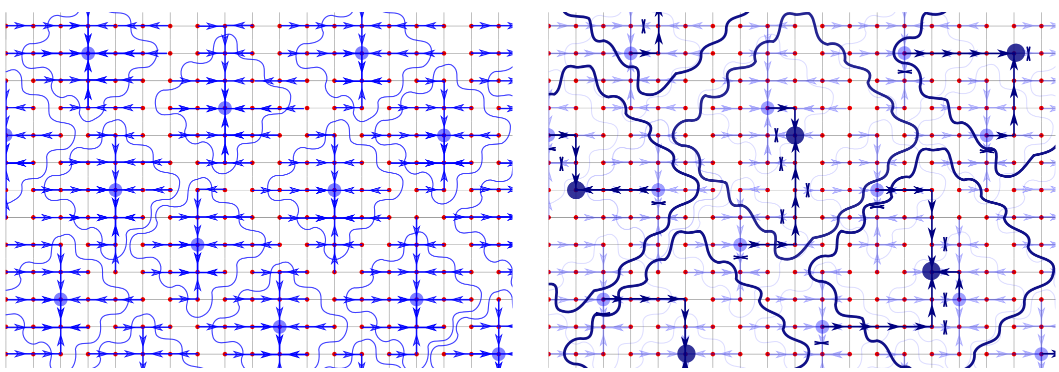

FIGURE 1. A_0 (red), A_1 (blue) and f_0 (blue arrows) – A_2 and f_1 (deep blue)

Observe that any measure preserving locally finite Borel graph with positive isoperimetric constant satisfies the hypotheses of Theorem 2.6 (any such graph in fact has exponential growth).

Proof of Theorem 2.6. Fix a sequence of natural numbers r_n, $n \geq 1$, such that $\sum_n 2r_{n+1}/(cr_n^2) < \infty$ (e.g., take $r_n = 2^n$). For each $n \geq 1$ let A_n be a Borel subset of X which is maximal with respect to the property that $B_{r_n}(x) \cap B_{r_n}(y) = \varnothing$ for all distinct $x, y \in A_n$. Then $\mu(A_n) \leq 1/(cr_n^2)$. Let $A_0 = X$. Define partial Borel functions f_n from X to X as follows. For each $x \in A_n$ choose the lexicographically least minimal length path $x = x_0, x_1, \ldots, x_k$ from x to an element of A_{n+1}. Let f_n be the function obtained by taking the union of all pairs (x_i, x_{i+1}) from these paths. See Figure 1. Observe that by maximality of A_{n+1}, the length of each such path

$x_0, x_1, \ldots, x_k$ is at most $2r_{n+1}$, and hence $\mu(\mathrm{dom}(f_n)) \leq 2r_{n+1}\mu(A_n) \leq 2r_{n+1}/(cr_n^2)$. Therefore, $\sum_n \mu(\mathrm{dom}(f_n))$ converges. The remaining properties from Lemma 2.5 are trivial to verify. $\square$

Remark 2.7. We note that a slightly refined argument yields the same result as Theorem 2.6 for graphs $\mathcal{G}$ of **superlinear growth**: for every $c > 0$ there exists an $r > 0$ such that for every $x \in X$, $B_r(x) \geq c|r|$. Since we will not need this, we omit the proof.

2.2. **μ-hyperfinite graphs.** We now turn to the case of μ-hyperfinite graphs. Suppose $\mathcal{G}$ is a locally finite Borel graph on a standard probability space (X, μ), and $\mathcal{G}$ is μ-hyperfinite. We do not assume that μ is preserved. Then after ignoring a null set we can find an acyclic Borel subgraph $\mathcal{T} \subseteq \mathcal{G}$ on the same vertex set and having the same connectedness relation. This subgraph will have at least as many ends as $\mathcal{G}$ in each connected component. Hence, on the set where $\mathcal{G}$ has more than two ends, we can apply Theorem 2.3 to $\mathcal{T}$ to find a one-ended spanning subforest.

Thus, it will suffice to handle μ-hyperfinite $\mathcal{G}$ that are a.e. one-ended. We begin with the following lemma:

Lemma 2.8. *Let $\mathcal{G}$ be a locally finite Borel graph on a standard probability space (X, μ) in which every connected component has one end. Let $\mathcal{T} \subseteq \mathcal{G}$ be an acyclic Borel subgraph with $\mathcal{R}_\mathcal{T} = \mathcal{R}_\mathcal{G}$ and assume that every connected component of $\mathcal{T}$ is a 2-regular line. Then there exists a Borel set $X' \subseteq X$ and an acyclic Borel subgraph $\mathcal{T}' \subseteq \mathcal{G}_{\restriction X'}$ such that*

1. *X' is a complete section for $\mathcal{R}_\mathcal{G}$ with $\mu(X') < 3/4$,*
2. *$\mathcal{R}_{\mathcal{T}'} = \mathcal{R}_{(\mathcal{G}_{\restriction X'})} = (\mathcal{R}_\mathcal{G})_{\restriction X'}$,*
3. *Every connected component of $\mathcal{G}_{\restriction X'}$ has one end, and every connected component of $\mathcal{T}'$ is a 2-regular line,*
4. *For $x, y \in X'$, we have $(x, y) \in \mathcal{T}'$ if and only if there is no $z \in X'$ in the interior of the line segment between x and y in $\mathcal{T}$.*

Proof. For $e = (x, y) \in \mathcal{G}$, let $[e]$ denote the set of points in X which lie strictly between x and y on $\mathcal{T}$ (so $x, y \notin [e]$, and $[e] = \varnothing$ whenever $e \in \mathcal{T}$). For each $e \in \mathcal{G}$ the set $\{e' = (u, v) \in \mathcal{G} : \text{either } u \in [e] \text{ or } v \in [e]\}$ is finite; define

$$n(e) = \max\{|[e']| : e' = (u, v) \in \mathcal{G} \text{ and either } u \in [e] \text{ or } v \in [e]\}.$$

For each $N \geq 1$ let $\mathcal{G}_N = \{e \in \mathcal{G} : |[e]| \leq N \text{ and } n(e) \leq N\}$. Then $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \cdots$ and $\bigcup_N \mathcal{G}_N = \mathcal{G}$. For a subset $\mathcal{G}'$ of $\mathcal{G}$ let $[\mathcal{G}'] = \bigcup\{[e] : e \in \mathcal{G}'\} \subseteq X$. Since $\mathcal{G}$ is one-ended we have that $[\mathcal{G}] = X$. Therefore, by choosing N large enough we may ensure that $\mu([\mathcal{G}_N]) > 3/4$.

Define a graph $\widetilde{\mathcal{H}}$ whose vertex set is $\mathcal{G}_N$ by $\widetilde{\mathcal{H}} = \{(e_0, e_1) \in \mathcal{G}_N \times \mathcal{G}_N : [e_0] \cap [e_1] \neq \varnothing\}$. The graph $\widetilde{\mathcal{H}}$ has bounded degree since if $e_0 \in \mathcal{G}_N$ then $|[e_0]| \leq N$ so there are less than $2N^2$ many (this is not exact) intervals in $\mathcal{T}$ of length at most N which intersect $[e_0]$. By [KST99, Proposition 4.6] we may therefore find a Borel coloring of the vertices of $\widetilde{\mathcal{H}}$ using only finitely many colors, say $C_1, \ldots, C_k$. Define now

$\mathcal{G}_N^0 = \mathcal{G}_N$ and for $1 \le i \le k$ define

$$\mathcal{G}_N^i = \mathcal{G}_N^{i-1} \setminus \{e \in C_i \,:\, [e] \subseteq [\mathcal{G}_N^{i-1} \setminus \{e\}]\}.$$

Then $[\mathcal{G}_N^k] = [\mathcal{G}_N]$, and $\widetilde{\mathcal{H}}_{\restriction \mathcal{G}_N^k}$ has degree bounded by 2, so by [KST99, Proposition 4.6] there is a Borel 3-coloring of $\widetilde{\mathcal{H}}_{\restriction \mathcal{G}_N^k}$, say with color sets D_0, D_1, and D_2. Then $[\mathcal{G}_N] = [\mathcal{G}_N^k] = [D_0] \cup [D_1] \cup [D_2]$, so there is some $i \in \{0,1,2\}$ with $\mu([D_i]) \ge \mu([\mathcal{G}_N])/3 > 1/4$.

We take $X' = X \setminus [D_i]$, so that $\mu(X') < 3/4$, and we let $\mathcal{T}' = (\mathcal{T}_{\restriction X'}) \cup D_{i \restriction X'}$. We have $\mathcal{T}' \subseteq \mathcal{G}_{\restriction X'}$, and $\mathcal{R}_{\mathcal{T}'} = (\mathcal{R}_{\mathcal{T}})_{\restriction X'}$ since D_i is an independent set for $\widetilde{\mathcal{H}}$. Property (2) easily follows. It is also clear that every connected component of $\mathcal{T}'$ is a 2-regular line, and that $(x,y) \in \mathcal{T}'$ if and only if there is no $z \in X'$ on the interior of the line segment between x and y in $\mathcal{T}$. This implies that for $e = (x,y) \in \mathcal{G}_{\restriction X'}$ we have $[e]^{\mathcal{T}'} = [e] \cap X'$, where $[e]^{\mathcal{T}'}$ denotes the set of points in X' which lie strictly between x and y in $\mathcal{T}'$. For (3), it remains to show that every connected component of $\mathcal{G}_{\restriction X'}$ has one end. Since $\mathcal{G}_{\restriction X'}$ is locally finite, this is equivalent to showing that for each $x \in X'$ the set $\{e \in \mathcal{G}_{\restriction X'} \,:\, x \in [e]^{\mathcal{T}'}\}$ is infinite. Since $\mathcal{G}$ is one-ended and locally finite, for each $x \in X$ the set $\{e \in \mathcal{G} \,:\, |[e]| > N \text{ and } x \in [e]\}$ is infinite. It therefore suffices to show that the set $\{e \in \mathcal{G} \,:\, |[e]| > N\}$ is contained in $\mathcal{G}_{\restriction X'}$ (since for $x \in X'$ and $e \in \mathcal{G}_{\restriction X'}$ we have $x \in [e]^{\mathcal{T}'}$ if and only if $x \in [e]$). If $|[e]| > N$ then neither endpoint of e lies in $[\mathcal{G}_N]$, so both endpoints of e lie in $X \setminus [\mathcal{G}_N] \subseteq X \setminus [D_i] = X'$, hence $e \in \mathcal{G}_{\restriction X'}$. Finally, X' is a complete section for $\mathcal{R}_{\mathcal{G}}$ since the set $\{e \in \mathcal{G} \,:\, |[e]| > N\}$ meets every connected component of $\mathcal{G}$ and, as we just showed, this set is contained in $\mathcal{G}_{\restriction X'}$. □

Lemma 2.9. *Let $\mathcal{G}$ be a locally finite Borel graph on a standard probability space (X,μ) in which every connected component has one end. Let $\mathcal{T}_0 \subseteq \mathcal{G}$ be an acyclic Borel subgraph with $\mathcal{R}_{\mathcal{T}_0} = \mathcal{R}_{\mathcal{G}}$ and assume that every connected component of $\mathcal{T}_0$ has 2 ends. Then there is a one-ended a.e. one-ended spanning subforest $\mathcal{T}$ of $\mathcal{G}$ such that $\mathcal{R}_{\mathcal{T}} = \mathcal{R}_{\mathcal{G}}$ μ-a.e.*

Proof. After moving to a finite-to-1 minor of $\mathcal{G}$, we may assume without loss of generality that every component of $\mathcal{T}_0$ is a 2-regular line. By Lemma 2.8, after ignoring a μ-null set we may find a vanishing sequence of Borel sets $X = X_0 \supseteq X_1 \supseteq \cdots$ along with acyclic subgraphs $\mathcal{T}_n \subseteq \mathcal{G}_{\restriction X_n}$ such that, for all $n \ge 0$:

(1) X_{n+1} is a complete section for $\mathcal{R}_{(\mathcal{G}_{\restriction X_n})}$,
(2) $\mathcal{R}_{\mathcal{T}_n} = \mathcal{R}_{(\mathcal{G}_{\restriction X_n})} = (\mathcal{R}_{\mathcal{G}})_{\restriction X_n}$,
(3) Every connected component of $\mathcal{G}_{\restriction X_n}$ has one end, and every connected component of $\mathcal{T}_n$ is a 2-regular line.
(4) For $x, y \in X_{n+1}$ we have $(x,y) \in \mathcal{T}_{n+1}$ if and only if there is no $z \in X_{n+1}$ in the interior of the line segment between x and y in $\mathcal{T}_n$.

Note that by (4) and induction on $n \ge 0$, for $x, y \in X_n$ we have $(x,y) \in \mathcal{T}_n$ if and only if there is no $z \in X_n$ in the interior of the line segment between x and y in $\mathcal{T}_0$. Now, for each $n \ge 0$, each component of $\mathcal{T}_{n \restriction (X_n \setminus X_{n+1})}$ is a finite line segment in $\mathcal{T}_n$. Let $s_n : X_n \setminus X_{n+1} \to X_{n+1}$ be any $\mathcal{T}_{n \restriction (X_n \setminus X_{n+1})}$-invariant

Borel function such that for each $x \in X_n$ the point $s_n(x) \in X_{n+1}$ which is $\mathcal{T}_n$-adjacent to the $\mathcal{T}_{n \upharpoonright (X_n \setminus X_{n+1})}$-component of x. Define now $f : X \to X$ as follows. For $x \in X_n \setminus X_{n+1}$ define $f(x) \in X_n$ be the $\mathcal{T}_n$-neighbor of x which lies along the unique $\mathcal{T}_n$-path from x to $s_n(x)$. Since the sets X_n are vanishing, this defines f on all of X. Clearly $(x, f(x)) \in \mathcal{G}$ for all $x \in X$, and $f^i(x) \neq x$ for all $i \geq 1$ (since $f^i(x)$ is eventually in X_n for any fixed n and large i). If $x \in X_0 \setminus X_{n+1}$, then the back-orbit of x under f is contained in the $\mathcal{T}_{0 \upharpoonright (X_0 \setminus X_{n+1})}$-component of x, which is finite. Thus, f defines a one-ended subforest of $\mathcal{G}$. It remains to show that any two $\mathcal{R}_\mathcal{G}$-related points eventually meet under f. By induction on $n \geq 0$, we see that each $\mathcal{T}_{0 \upharpoonright (X_0 \setminus X_{n+1})}$-component contains exactly one $\mathcal{T}_{n \upharpoonright (X_n \setminus X_{n+1})}$-component, and that if two points $x, y \in X_0 \setminus X_{n+1}$ are in the same $\mathcal{T}_{0 \upharpoonright (X_0 \setminus X_{n+1})}$-component, then there exists $i, j > 0$ such that $f^i(x) = s_n \circ \cdots \circ s_0(x) = s_n \circ \cdots \circ s_0(y) = f^j(y)$. ☐

We are now finished in the hyperfinite case.

Lemma 2.10. *Let $\mathcal{G}$ be a locally finite Borel graph on a standard probability space (X, μ). Assume that $\mathcal{G}$ is μ-hyperfinite but μ-nowhere 2-ended. Then $\mathcal{G}$ has a μ-a.e.one-ended spanning subforest $\mathcal{T}$. Moreover, if $\mathcal{G}$ is measure-preserving, then we may choose $\mathcal{T}$ so that $\mathcal{R}_\mathcal{T} = \mathcal{R}_\mathcal{G}$ μ-a.e.*

Proof. Since $\mathcal{G}$ is μ-hyperfinite, after ignoring a $\mathcal{G}$-invariant μ-null set we may find an acyclic Borel graph $\mathcal{T}_0 \subseteq \mathcal{G}$ with $\mathcal{R}_{\mathcal{T}_0} = \mathcal{R}_\mathcal{G}$. Let X_1 be the ($\mathcal{R}_\mathcal{G}$-invariant) set of vertices x so that $\mathcal{T}_{0 \upharpoonright [x]_{\mathcal{R}_{\mathcal{T}_0}}}$ is one-ended, X_2 similarly for the components on which $\mathcal{T}_0$ is two-ended, and finally let X_3 be the set of components on which $\mathcal{T}_0$ has more than two ends. $\mathcal{T}_{0 \upharpoonright X_1}$ is a Borel a.e. one-ended spanning subforest of $\mathcal{G}_{\upharpoonright X_1}$. The number of ends of $\mathcal{G}_{\upharpoonright X_2}$ is one since it is bounded above by the number of ends of $\mathcal{T}_{0 \upharpoonright X_2}$. By Lemma 2.9 we can find a Borel a.e. one-ended spanning subforest of $\mathcal{G}_{\upharpoonright X_2}$. By Theorem 2.3 we can find a Borel a.e. one-ended spanning subforest of $\mathcal{T}_{0 \upharpoonright X_3}$ and thus of $\mathcal{G}_{\upharpoonright X_3}$.

Finally, note that in the case when $\mathcal{G}$ is measure preserving, $\mathcal{T}_0$ must have either one or two end almost everywhere, and so $\mathcal{R}_\mathcal{T} = \mathcal{R}_\mathcal{G}$ by the condition that $\mathcal{R}_\mathcal{T} = \mathcal{R}_\mathcal{G}$ in Lemma 2.9. ☐

Let $\mathcal{G}$ be a locally finite Borel graph on a standard probability space (X, μ). If $\mathcal{S} \subseteq \mathcal{R}_\mathcal{G}$ is an equivalence relation, then we say that $\mathcal{S}$ is **$\mathcal{G}$-connected** if $\mathcal{S} \cap \mathcal{G}$ is a graphing of $\mathcal{S}$. Consider the collection

$$\mathrm{CH}(\mathcal{G}) = \{\mathcal{S} \subseteq \mathcal{R}_\mathcal{G} : \mathcal{S} \text{ is a } \mathcal{G}\text{-connected } \mu\text{-hyperfinite equivalence relation}\}$$

Since an increasing union of $\mathcal{G}$-connected equivalence relations is $\mathcal{G}$-connected, and an increasing union of μ-hyperfinite equivalence relations is μ-hyperfinite, by measure theoretic exhaustion we see that any $\mathcal{S}_0 \in \mathrm{CH}(\mathcal{G})$ is contained in a μ-maximal element $\mathcal{S}$ of $\mathcal{C}$ (i.e., if $\mathcal{Q} \in \mathcal{C}$ satisfies $[x]_\mathcal{S} \subseteq [x]_\mathcal{Q}$ for μ-a.e. $x \in X$, then in fact $[x]_\mathcal{S} = [x]_\mathcal{Q}$ for μ-a.e. $x \in X$).

We now have the following two corollaries.

Corollary 2.11. *Let $\mathcal{G}$ be a p.m.p. locally finite Borel graph on (X, μ). Assume that $\mathcal{G}$ is aperiodic and μ-nowhere two-ended. Then $\mathcal{G}$ admits a μ-a.e. one-ended Borel spanning subforest $\mathcal{T} \subseteq \mathcal{G}$ such that $\mathcal{R}_{\mathcal{T}}$ is a μ-maximal element of* $\mathrm{CH}(\mathcal{G})$.

Observe that the preservation of the measure is a necessary condition here since, for instance, the free group $\mathbf{F}_2$ admits μ-hyperfinite non-singular free actions $\mathbf{F}_2 \curvearrowright (X, \mu)$. The treeing $\mathcal{G}$ associated with a free generating set is the single μ-maximal element of $\mathrm{CH}(\mathcal{G})$.

Proof. Since $\mathcal{G}$ is p.m.p., aperiodic, and μ-nowhere two-ended, $\mathcal{G}$ admits a μ-a.e. one-ended Borel spanning subforest $\mathcal{T}_0 \subseteq \mathcal{G}$. Let $\mathcal{S}$ be a μ-maximal element of $\mathrm{CH}(\mathcal{G})$ containing $\mathcal{R}_{\mathcal{T}_0}$, and let $\mathcal{G}_0 = \mathcal{S} \cap \mathcal{G}$, so that $\mathcal{G}_0$ is a graphing of $\mathcal{S}$. Since $\mathcal{G}_0$ is p.m.p. and μ-hyperfinite, almost every connected component of $\mathcal{G}_0$ has at most two ends. Since $\mathcal{G}_0$ contains $\mathcal{T}_0$ as a μ-a.e. one-ended subforest, $\mathcal{G}_0$ must be μ-nowhere two-ended (by Lemma 2.4), hence $\mathcal{G}_0$ is μ-a.e. one-ended. Now (by hyperfiniteness of $\mathcal{G}_0$) after discarding a nullset we can find a acyclic Borel graph $\mathcal{T}' \subseteq \mathcal{G}_0$ such that $\mathcal{R}_{\mathcal{T}'} = \mathcal{R}_{\mathcal{G}}$. To finish, apply Lemma 2.9 to obtain a μ-a.e. one-ended Borel subtreeing $\mathcal{T}$ of $\mathcal{R}_{\mathcal{G}}$ which will satisfy the conclusion of the theorem. $\square$

Corollary 2.12. *Let $\mathcal{G}$ be a locally finite Borel graph on (X, μ) which is μ-nowhere two-ended. Assume that there exists an acyclic Borel subgraph $\mathcal{T} \subseteq \mathcal{G}$ with $\mathcal{R}_{\mathcal{T}} = \mathcal{R}_{\mathcal{G}}$ modulo μ. Then $\mathcal{G}$ has a Borel a.e. one-ended spanning subforest.*

Proof. Let X_0 be the $\mathcal{G}$-invariant set consisting of points whose $\mathcal{T}$-connected component has two ends. We can find a Borel a.e. one-ended spanning subforest of $\mathcal{G}_{\restriction (X \setminus X_0)}$ by Theorem 2.3. Note that $\mathcal{G}_{\restriction X_0}$ is a.e. one-ended because $\mathcal{T}$ and $\mathcal{G}$ have the same connectedness relation, so each connected component of $\mathcal{T}$ has at least as many ends as $\mathcal{G}$, and $\mathcal{G}$ is μ-nowhere two-ended. Thus, we can find a Borel a.e. one-ended spanning subforest of $\mathcal{G}_{\restriction X_0}$ by Lemma 2.9. $\square$

2.3. **Proof of Theorem 2.1.** We recall one final lemma from [CMTD16].

Lemma 2.13 ([CMTD16, Proposition 3.1])**.** *Let $\mathcal{G}$ be a locally finite Borel graph on a standard Borel space X. Let X_0 be a Borel subset of X which meets every connected component $\mathcal{G}$, and suppose that $\mathcal{G}_{\restriction X_0}$ has a Borel one-ended spanning subforest $\mathcal{H}_0$. Then $\mathcal{H}_0$ can be extended to a Borel one-ended spanning subforest $\mathcal{H}$ of $\mathcal{G}$.*

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Supposing $\mathcal{G}$ is a locally finite Borel graph on a standard probability space (X, μ), by an exhaustion argument, we can find a Borel partition of X into countably many Borel $\mathcal{G}$-invariant sets $A, B_0, B_1, \ldots$ such that $\mathcal{G}_{\restriction A}$ is μ-hyperfinite, and for each $i \geq 0$ there is a Borel set $C_i \subseteq B_i$ which meets every connected component of $\mathcal{G}_{\restriction B_i}$ such that $\mathcal{G}_{\restriction C_i}$ has positive isoperimetric constant.

We can find a Borel a.e. one-ended spanning subforest of $\mathcal{G}_{\restriction A}$ by Lemma 2.10. We can similarly find a Borel a.e. one-ended spanning subforest of each $\mathcal{G}_{\restriction C_i}$ by Theorem 2.6. These subforests can be extended to Borel a.e. one-ended subforests of $\mathcal{G}_{\restriction B_i}$ by Lemma 2.13. $\square$

3. Planar graphs are measure treeable

In this section we show that graphs which are Borel planar (see Definition 3.5) are measure treeable. This will imply that these graphs admit Borel a.e. one-ended spanning subforests with respect to any Borel probability measure, provided the graph is μ-nowhere two-ended (Corollary 3.7).

3.1. **Preliminaries.** A **2-basis** for a (undirected multi-)graph $\mathcal{G}$ is a collection $\mathcal{B}$ of simple cycles in $\mathcal{G}$ such that
(1) no edge of $\mathcal{G}$ is contained in more than 2 cycles in $\mathcal{B}$, and
(2) $\mathcal{B}$ generates the cycle space of $\mathcal{G}$, i.e., for every cycle C of $\mathcal{G}$ there exists $B_0, \ldots, B_{n-1} \in \mathcal{B}$ with $1_C = \sum_{i<n} 1_{B_i} \pmod 2$ (we identify each cycle with its unordered set of edges).

Assume for the moment that $\mathcal{G}$ is locally finite and 2-vertex-connected. Let $\widehat{\mathcal{G}}$ be a planar embedding of $\mathcal{G}$, and assume that $\widehat{\mathcal{G}}$ is **accumulation-free**, i.e., every compact subset of the plane meets only finitely many vertices and edges of $\widehat{\mathcal{G}}$. A **face** of $\widehat{\mathcal{G}}$ is a connected component of the complement of $\widehat{\mathcal{G}}$. The boundary of a face of $\widehat{\mathcal{G}}$ is either a finite cycle or a bi-infinite 2-regular path. In the former case, the boundary cycle is called a **facial cycle**. The following theorem of Thomassen relates 2-bases to accumulation-free planar embeddings.

Theorem 3.1 (Thomassen [Tho80])**.** *Let $\mathcal{G}$ be a 2-vertex-connected locally finite graph.*

(1) If $\widehat{\mathcal{G}}$ is an accumulation-free planar embedding of $\mathcal{G}$, then the set $\mathcal{B}$ of facial cycles of $\widehat{\mathcal{G}}$ forms a 2-basis for $\mathcal{G}$.
(2) If $\mathcal{G}$ admits a 2-basis $\mathcal{B}$, then $\mathcal{G}$ has an accumulation-free planar embedding $\widehat{\mathcal{G}}$ in which $\mathcal{B}$ is the set of facial cycles of $\widehat{\mathcal{G}}$.

In general, if a graph $\mathcal{G}$ is not necessarily 2-vertex-connected, then we can consider **blocks** of $\mathcal{G}$, i.e., maximal 2-vertex-connected subsets of $\mathcal{G}$. Then a set $\mathcal{B}$ of cycles of $\mathcal{G}$ is a 2-basis for $\mathcal{G}$ if and only if the restriction of $\mathcal{B}$ to each block of $\mathcal{G}$ is a 2-basis for that block. Furthermore, the incidence relation on blocks gives an acyclic graph structure on the set of blocks.

Let $\mathcal{B}$ be a 2-basis for $\mathcal{G}$. The **dual** of $\mathcal{G}$ (with respect to $\mathcal{B}$) is the undirected (multi-)graph $\mathcal{G}^*$, with vertex set $\mathcal{B}$, and defined as follows. Let $\mathcal{G}_2 \subseteq \mathcal{G}$ be the set of edges of $\mathcal{G}$ which belong to exactly two cycles from $\mathcal{B}$, and for each edge $e \in \mathcal{G}_2$ introduce a corresponding edge $e^* \in \mathcal{G}^*$ which connects B_0 with B_1, where $B_0, B_1 \in \mathcal{B}$ are the unique cycles for which $e \in B_0 \cap B_1$. Thus $\mathcal{G}^* = \{e^* : e \in \mathcal{G}_2\}$, and two distinct cycles $B_0, B_1 \in \mathcal{B}$ are connected by one edge e^* for each edge $e \in B_0 \cap B_1$. The connected components of $\mathcal{G}^*$ correspond precisely to blocks of $\mathcal{G}$, i.e., two elements of $\mathcal{B}$ are in the same connected component of $\mathcal{G}^*$ if and only if they are contained in the same block of $\mathcal{G}$.

Part (2) of Theorem 3.1 easily implies the following.

Proposition 3.2. *Let $\mathcal{G}$ be a locally finite graph on X which is 2-vertex-connected. Let $\mathcal{B}$ be a 2-basis for $\mathcal{G}$ and assume that every edge of $\mathcal{G}$ belongs to two distinct* *cycles from $\mathcal{B}$. Let $\mathcal{G}^*$ be the dual of $\mathcal{G}$ with respect to $\mathcal{B}$. For each $x \in X$ let $x^* = \{e^* : e \text{ is incident with } x\}$. Then $X^* = \{x^* : x \in X\}$ is a 2-basis for $\mathcal{G}^*$, every edge of $\mathcal{G}^*$ belongs to two distinct elements of X^*, and the map $x \mapsto x^*$ provides an isomorphism between $\mathcal{G}$ and $\mathcal{G}^{**}$.*

Given a 2-basis $\mathcal{B}$ of a planar graph $\mathcal{G}$, together with a subgraph $\mathcal{G}_0^*$ of the dual $\mathcal{G}^*$, we define $\mathcal{G} \mathbin{\overset{*}{\ominus}} \mathcal{G}_0^*$ to be the subgraph of $\mathcal{G}$ obtain by removing all edges which cross an edge of $\mathcal{G}_0^*$, i.e.,

$$\mathcal{G} \mathbin{\overset{*}{\ominus}} \mathcal{G}_0^* = \mathcal{G} \setminus \{e \in \mathcal{G} : e^* \in \mathcal{G}_0^*\}$$

(to be pronounced "o minus star" or "cyclope"). The following proposition explains how various properties of $\mathcal{G}_0^*$ are reflected in $\mathcal{G} \mathbin{\overset{*}{\ominus}} \mathcal{G}_0^*$.

Proposition 3.3. *Let $\mathcal{G}$ be an aperiodic locally finite graph on X which is 2-vertex-connected. Let $\mathcal{B}$ be a 2-basis for $\mathcal{G}$ and assume that every edge of $\mathcal{G}$ belongs to two distinct cycles from $\mathcal{B}$. Let $\mathcal{G}^*$ be the dual of $\mathcal{G}$ with respect to $\mathcal{B}$. Let $\mathcal{G}_0^*$ be a spanning subgraph of $\mathcal{G}^*$, and let $\mathcal{H} = \mathcal{G} \mathbin{\overset{*}{\ominus}} \mathcal{G}_0^*$. Then:*

(1) $\mathcal{H}$ is acyclic if and only if $\mathcal{G}_0^*$ is aperiodic;
(2) $\mathcal{H}$ is aperiodic if and only if $\mathcal{G}_0^*$ is acyclic;
(3) $\mathcal{H}$ is acyclic with the same connected components as $\mathcal{G}$ if and only if $\mathcal{G}_0^*$ is a one-ended spanning subforest of $\mathcal{G}^*$.

Proof. Since $\mathcal{G}$ is 2-vertex-connected, every connected component of $\mathcal{G}$ corresponds to a single component of $\mathcal{G}^*$.

We may therefore assume that $\mathcal{G}$ has a single connected component, and hence that $\mathcal{G}^*$ is connected. By Theorem 3.1, there is an accumulation-free planar embedding $\widehat{\mathcal{G}}$ of $\mathcal{G}$ such that $\mathcal{B}$ is the set of facial cycles of $\widehat{\mathcal{G}}$. Let F be a face of $\widehat{\mathcal{G}}$. Then the boundary of F is a finite cycle $B_F \in \mathcal{B}$ (as opposed to a bi-infinite line) and the closure of F is compact since otherwise the complement of F would be a compact set containing all of $\widehat{\mathcal{G}}$, contradicting that $\mathcal{G}$ is aperiodic and the embedding is accumulation-free.

(1): Suppose that $\mathcal{G}_0^*$ is aperiodic. Let C be a simple cycle in $\widehat{\mathcal{G}}$ and let F_0 be a face of $\widehat{\mathcal{G}}$ in the interior of C. Since the interior of C is precompact and the embedding is accumulation-free, there must be some face F_1 in the exterior of C such that B_{F_0} and B_{F_1} are connected by a path through $\mathcal{G}_0^*$. This path must contain some edge from $\{e^* : e \in C\}$, and thus C is not contained in $\mathcal{H}$. This shows that $\mathcal{H}$ is acyclic. Conversely, suppose that $\mathcal{G}_0^*$ has a finite connected component $\mathcal{B}_0 \subseteq \mathcal{B}$. Then $\mathcal{K}_0 = \bigcup \mathcal{B}_0$ is a finite connected subgraph of $\mathcal{G}$, so the image $\widehat{\mathcal{K}}_0$ of $\mathcal{K}_0$ has a unique face F which contains infinitely many vertices of $\widehat{\mathcal{G}}_0$. The boundary of F is a cycle $C \subseteq \mathcal{K}_0$, and if $e \in C$ then e is contained in only one cycle from $\mathcal{B}_0$ and hence $e^* \notin \mathcal{G}_0^*$. This shows that C is a cycle in $\mathcal{H}$.

(2): This follows from (1) and Proposition 3.2.

(3): Suppose that $\mathcal{G}_0^*$ is a one-ended spanning subforest of $\mathcal{G}^*$. By part (1), $\mathcal{H}$ is acyclic, so it remains to show that $\mathcal{H}$ is connected. Suppose that $x, y \in X$ are $x, y \in X$ are connected by an edge $e_0 \in \mathcal{G}$, but are not connected by an edge in $\mathcal{H}$. This means that $e_0^* \in \mathcal{G}_0^*$. Since $\mathcal{G}_0^*$ is a one-ended forest, there is a there is a unique finite connected component of $\mathcal{G}_0^*\setminus\{e_0^*\}$; let $\mathcal{B}_0$ be its set of vertices. Then the set $\{e \in \mathcal{G} : e \neq e_0 \text{ and } e \text{ is in exactly one } B \in \mathcal{B}_0\}$ is a path from x to y through $\mathcal{H}$, hence $\mathcal{H}$ is connected. For the converse, by Proposition 3.2, we may assume that $\mathcal{G}_0^*$ is acyclic with the same connected components as $\mathcal{G}^*$, toward the goal of showing that $\mathcal{H}$ is a one-ended spanning subforest of $\mathcal{G}$. If $\mathcal{H}$ had more than one end, then $\mathcal{H}$ would contain a bi-infinite line whose image in $\widehat{\mathcal{G}}$ divides the plane into two connected components. Then there would be no path through $\mathcal{G}_0^*$ connecting cycles which lie on opposite sides of this bi-infinite line through $\mathcal{H}$, contrary to $\mathcal{G}_0^*$ being connected. □

3.2. **Borel planar graphs.** Our idea now is to use Lemma 2.13, along with part (3) of Proposition 3.3 to obtain acyclic subgraphs of graphs which admit a Borel 2-basis. To handle some technicalities, we will need the following Proposition.

Proposition 3.4 ([CMTD16])**.** *Suppose that $\mathcal{G}$ is a locally finite Borel graph on X, and $A \subseteq X$ is Borel. Then there is an acyclic Borel function $f : [A]_{\mathcal{R}_\mathcal{G}} \setminus A \to [A]_{\mathcal{R}_\mathcal{G}}$ whose back-orbits are all finite, and whose graph is contained in $\mathcal{G}$.*

Definition 3.5 (Borel planar graphs)**.** Let $\mathcal{G}$ be a locally finite Borel graph on a standard Borel space X. A **Borel 2-basis** for $\mathcal{G}$ is a 2-basis $\mathcal{B}$ for $\mathcal{G}$ which is Borel when viewed as a subset of the standard Borel space of all finite subsets of $\mathcal{G}$. We say that $\mathcal{G}$ is **Borel planar** if it admits a Borel 2-basis.

Observe that if a locally finite Borel graph $\mathcal{G}$ admits a Borel 2-basis $\mathcal{B}$, then the dual graph $\mathcal{G}^*$ is a Borel graph on $\mathcal{B}$. We can now prove that Borel graphs which admit a Borel 2-basis are measure treeable.

Theorem 3.6. *Let $\mathcal{G}$ be a locally finite Borel planar graph on X. Let μ be a Borel probability measure on X. Then there exists a $\mathcal{G}$-invariant μ-conull Borel set $X_0 \subseteq X$ and an acyclic Borel subgraph $\mathcal{G}_0 \subseteq \mathcal{G}$ with $(\mathcal{R}_{\mathcal{G}_0})_{\restriction X_0} = (\mathcal{R}_\mathcal{G})_{\restriction X_0}$.*

Proof. It suffices to produce an acyclic Borel subgraph of $\mathcal{G}$ whose restriction to almost every block is connected. Thus, by working block by block we may assume without loss of generality that each component of $\mathcal{G}$ is 2-vertex-connected.

Fix a Borel 2-basis $\mathcal{B}$ for $\mathcal{G}$ and let $\mathcal{G}^*$ be the dual graph on $\mathcal{B}$. For the rest of the proof, elements of $\mathcal{B}$ will be referred to as **facial cycles**. Let $\mathcal{G}_1$ be the set of edges of $\mathcal{G}$ which are contained in exactly one facial cycle. Let $\mathcal{B}_1 \subseteq \mathcal{B}$ be the set of all facial cycles which are incident with an edge from $\mathcal{G}_1$. Assume first that $\mathcal{G}_1$ meets every connected component of $\mathcal{G}$. Then, since $\mathcal{G}$ is 2-vertex-connected, $\mathcal{B}_1$ meets every connected component of $\mathcal{G}^*$. Apply Proposition 3.4 to the graph $\mathcal{G}^*$ and the set $\mathcal{B}_1$ to obtain an acyclic subgraph $\mathcal{G}_0^*$ of $\mathcal{G}^*$ in which every connected component either is one-ended and does not meet $\mathcal{B}_1$, or is finite and meets $\mathcal{B}_1$ in exactly one point. Let $B \mapsto e(B)$ be a Borel function selecting one edge $e(B) \in B \cap \mathcal{G}_1$ out of each $B \in \mathcal{B}_1$. Then arguing as in Proposition 3.3, we see that the graph $\mathcal{G} \circledast \big(\mathcal{G}_0^* \cup \{e(B) : B \in \mathcal{B}_1\}\big)$ is acyclic with the same connected components as $\mathcal{G}$, so the proof is complete in this case.

We may therefore assume for the rest of the argument that $\mathcal{G}_1 = \varnothing$. We may also assume that no connected component of $\mathcal{G}$ is acyclic. It follows that every edge of

$\mathcal{G}$ is contained in two distinct facial cycles. This implies that both $\mathcal{G}$ and $\mathcal{G}^*$ are everywhere one-ended.

If we can show that $\mathcal{G}^*$ has a Borel a.e. one-ended spanning subforest $\mathcal{T}^*$ then we will be done, since then by Proposition 3.3, the subgraph $\mathcal{G} \circledast \mathcal{T}^*$ of $\mathcal{G}$ would have the desired properties.

Toward this goal, choose, by the discussion preceding Corollary 2.11, an acyclic aperiodic hyperfinite Borel subgraph $\mathcal{T}$ of $\mathcal{G}$.

Let Y denote the $\mathcal{R}_{\mathcal{G}}$-saturation of the set where $\mathcal{T}$ does not have two ends. Theorem 2.3 and Lemma 2.13 imply that $\mathcal{G}_{\restriction Y}$ has a Borel a.e. one-ended spanning subforest $\mathcal{T}_0$. Then by Proposition 3.3, the graph $\mathcal{L} = (\mathcal{G}_{\restriction Y})^* \circledast \mathcal{T}_0$ is an acyclic subgraph of $(\mathcal{G}_{\restriction Y})^*$ with $\mathcal{R}_{\mathcal{L}} = \mathcal{R}_{(\mathcal{G}_{\restriction Y})^*}$ modulo μ. While $\mathcal{L}$ may have some components with two ends, every component of $\mathcal{G}^*$ is one-ended, so we can apply Corollary 2.12 to find a Borel a.e. one-ended spanning subforest of $(\mathcal{G}_{\restriction Y})^*$. By restricting our attention now to the complement of Y, we may assume without loss of generality that every connected component of $\mathcal{T}$ has two ends.

Let $\mathcal{H}^*$ be the subgraph of $\mathcal{G}^*$ given by $\mathcal{H}^* = \mathcal{G}^* \circledast \mathcal{T}$. Then $\mathcal{H}^*$ is an acyclic aperiodic subgraph of $\mathcal{G}^*$ by Proposition 3.3. Let $Z^* \subseteq \mathcal{B}$ denote the $\mathcal{R}_{\mathcal{G}^*}$-saturation of the set where $\mathcal{H}^*$ does not have two ends, and let $Z \subseteq X$ denote the set of vertices incident with some facial cycle in Z^*. Applying Theorem 2.3 and Lemma 2.13 again (but this time applied to $\mathcal{G}^*$), we see that $\mathcal{G}^*_{\restriction Z^*} = (\mathcal{G}_{\restriction Z})^*$ has a Borel a.e. one-ended spanning subforest. Thus, by restricting our attention to the complement of Z, we may assume without loss of generality that every connected component of $\mathcal{H}^*$ and every connected component of $\mathcal{T}$ has two ends.

Notice that if $B \in \mathcal{B}$ is a facial cycle of $\mathcal{G}$, then B cannot intersect more than 2 connected components of $\mathcal{T}$, since otherwise B would correspond to a "branch vertex" of $\mathcal{H}^*$ and hence $\mathcal{H}^*$ would have a connected component with more than 2 ends (i.e., if B intersects 3 components C_0, C_1, and C_2, of $\mathcal{T}$ then there is a path from B to infinity through $\mathcal{H}^*$ which leaves B through an edge between C_0 and C_1, and likewise for an edge between C_1 and C_2 and an edge between C_2 and C_0, so these three paths to infinity correspond to 3 different ends in the $\mathcal{H}^*$-connected component of B), contrary to our assumption.

Consider the set $\mathcal{C} \subseteq \mathcal{B}$ of facial cycles which intersect exactly 1 connected component of $\mathcal{T}$. Each infinite connected component of $\mathcal{H}^*_{\restriction\mathcal{C}}$ must have one-end, since if it had more than one end then some of these facial cycles would intersect more than 1 component of $\mathcal{T}$ (indeed, each bi-infinite line in $\mathcal{H}^*$ separates the plane into two components, and each facial cycle on such a line contains vertices in both components, and hence these vertices belong to distinct components of $\mathcal{T}$). So the $\mathcal{R}_{\mathcal{G}^*}$-saturation of the set where $\mathcal{H}^*_{\restriction\mathcal{C}}$ is infinite admits a Borel a.e. one-ended subforest.

We can therefore assume without loss of generality that $\mathcal{H}^*_{\restriction\mathcal{C}}$ has finite connected components. From here we will now show that $\mathcal{R}_{\mathcal{G}}$ is μ-hyperfinite.

Let $\mathcal{G}_0$ be obtained from $\mathcal{G}$ by removing every edge $e \in \mathcal{G} \setminus \mathcal{T}$ that belongs to some facial cycle in $\mathcal{C}$, and observe that the graphs $\mathcal{G}_0$ and $\mathcal{G}$ have the same connected components. This process, of removing edges from $\mathcal{G}$ to obtain $\mathcal{G}_0$, merges finite collections of faces (corresponding to components of $\mathcal{H}^*_{\restriction\mathcal{C}}$) with a unique face of in

$\mathcal{B} \setminus \mathcal{C}$, and produces a 2-basis $\mathcal{B}_0$ for $\mathcal{G}_0$. Now every $B \in \mathcal{B}_0$ intersects exactly two connected components of $\mathcal{T}$. The dual graph $\mathcal{G}_0^*$, of $\mathcal{G}_0$ with respect to $\mathcal{B}_0$, is a minor of $\mathcal{G}^*$ and, if we define $\mathcal{H}_0^* = \mathcal{G}_0^* \overset{*}{\ominus} \mathcal{T}$, then every connected component of $\mathcal{H}_0^*$ is a bi-infinite line. Therefore, after applying the analogous argument as well to the dual, we may assume without loss of generality that every connected component of $\mathcal{H}^*$ and of $\mathcal{T}$ is a bi-infinite line.

Now, every line in $\mathcal{T}$ intersects exactly two lines in $\mathcal{H}^*$, and likewise, every line in $\mathcal{H}^*$ intersects exactly two lines in $\mathcal{T}$, so the lines of $\mathcal{T}$ within any fixed $\mathcal{R}_{\mathcal{G}}$-class are themselves arranged in the structure of a single bi-infinite line (i.e., take two lines of $\mathcal{T}$ to be adjacent if they both intersect the same line in $\mathcal{H}^*$). This easily implies that $\mathcal{R}_{\mathcal{G}}$ is 2-amenable in the sense of Jackson-Kechris-Louveau [JKL02] (the proof of this is the same as Example 2.19 of their paper, showing that a scattered linear order of Hausdorff rank 2 is 2-amenable). Hence, by [JKL02] and [CFW81], $\mathcal{R}_{\mathcal{G}}$ is μ-hyperfinite. The conclusion of the theorem now follows. □

Corollary 3.7. *Let $\mathcal{G}$ be a locally finite Borel planar graph on X. Let μ be a Borel probability measure on X. Assume that $\mathcal{G}$ is μ-nowhere two-ended. Then $\mathcal{G}$ has a Borel a.e. one-ended spanning subforest.*

Proof. This follows from Theorem 3.6 and Corollary 2.12. □

We also note the following variation of Theorem 3.6.

Theorem 3.8. *Let $\mathcal{G}$ be a locally finite Borel planar graph on (X, μ), let $\mathcal{B}$ be a Borel 2-basis for $\mathcal{G}$, and let $\mathcal{G}^*$ be the associated dual graph. Let $\mathcal{H} \subseteq \mathcal{G}$ denote the set of edges which belong to at most one $B \in \mathcal{B}$. Assume that $\mathcal{G}^*$ is μ-nowhere two-ended. Then there exists a $\mathcal{G}$-invariant μ-conull Borel set $X_0 \subseteq X$ and an acyclic Borel graph $\mathcal{G}_0$ with $\mathcal{H} \subseteq \mathcal{G}_0 \subseteq \mathcal{G}$ with $(\mathcal{R}_{\mathcal{G}_0})_{\restriction X_0} = \mathcal{R}_{\mathcal{G}_{\restriction X_0}}$.*

Proof. Let Y denote the set of points $y \in X$ which are not incident to any edge in $\mathcal{H}$, and for $y \in Y$ let $y^* = \{e^* : e \text{ is incident to } y\}$. Then the set $Y^* = \{y^* : y \in Y\}$ is a Borel 2-basis for $\mathcal{G}^*$. Therefore, by Corollary 3.7, $\mathcal{G}^*$ has a Borel a.e. one-ended spanning subforest $\mathcal{T}$. Then the graph $\mathcal{G}_0 = \mathcal{G} \overset{*}{\ominus} \mathcal{T}$ is the desired subgraph. □

4. Measure strong treeability of $\mathrm{Isom}(\mathbf{H}^2)$

Throughout this section, we let G denote the group $\mathrm{Isom}(\mathbf{H}^2)$ of all isometries of the hyperbolic plane $\mathbf{H}^2 = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$, and we let K denote the stabilizer of $i \in \mathbf{H}^2$. Then K is a compact subgroup of G. The term cross-section is defined in the Appendix A.

Theorem 4.1. *The group $G = \mathrm{Isom}(\mathbf{H}^2)$ is measure strongly treeable. That is, given any free Borel action $G \curvearrowright X$ of G on a standard Borel space X and a Borel probability measure μ on X, there exists a G-invariant μ-conull Borel set $X_0 \subseteq X$ such that $(\mathcal{R}_G)_{\restriction Y}$ is Borel treeable for any Borel cross section $Y \subseteq X_0$.*

Proof. Let $G \curvearrowright X$ be a free Borel action of G on a standard Borel space X, and let μ be a Borel probability measure on X, which we may assume is $\mathcal{R}_G$-quasi-invariant. $\mathcal{R}_G$ descends to a Borel equivalence relation $\widehat{R}_G$ on $K \backslash X$. Define $d : \widehat{R}_G \to [0, \infty)$

by $d(K \cdot x,\, Kg \cdot x) = d_{\mathbf{H}^2}(g^{-1}(i), i)$, where $d_{\mathbf{H}^2}$ denotes the hyperbolic metric on $\mathbf{H}^2$. Then d is a Borel function on $\widehat{R}_G$ and the restriction of d to each $\widehat{R}_G$-class is a metric (since G acts by isometries on $\mathbf{H}^2$) which is isometrically isomorphic with $(\mathbf{H}^2, d_{\mathbf{H}^2})$.

Let $U \subseteq G$ be an open subset of G whose closure is compact and with $K \subseteq U$. Applying Theorem A.3 to the closure of U, we may find a Borel cross section $Y \subseteq X$ for the action which is cocompact and satisfies $U \cdot y_0 \cap U \cdot y_1 = \varnothing$ for distinct $y_0, y_1 \in Y$. Let ν be an $(\mathcal{R}_G)_{\restriction Y}$-quasi-invariant Borel probability measure on Y given by Proposition A.7, coming from the canonical $(\mathcal{R}_G)_{\restriction Y}$-quasi-invariant measure class associated to Y and μ. We will be done once we show that $(\mathcal{R}_G)_{\restriction Y}$ is treeable on a ν-conull invariant Borel set Y_0, since then the $\mathcal{R}_G$-saturation of Y_0 will be the μ-conull Borel set X_0 which satisfies the conclusion of the theorem.

Let $\widehat{Y}$ and $\widehat{\nu}$ denote the images of Y and ν respectively in $K\backslash X$. For each $\widehat{y} \in \widehat{Y}$ let $D(\widehat{y}) \subseteq [\widehat{y}]_{\widehat{R}_G}$ denote the corresponding Dirichlet region in the metric space $([\widehat{y}]_{\widehat{R}_G}, d_{\restriction [\widehat{y}]_{\widehat{R}_G}})$, i.e., $D(\widehat{y})$ consists of all points of $[\widehat{y}]_{\widehat{R}_G}$ whose distance to $\widehat{y}$ is strictly less than their distance to any other point of $\widehat{Y} \cap [\widehat{y}]_{\widehat{R}_G}$. Since G acts properly on $\mathbf{H}^2$ and $K \subseteq U$, there exists an $r_0 > 0$ such that $d_{\mathbf{H}^2}(g^{-1}(i), i) > r_0$ for all $g \notin U$. Therefore, for each $\widehat{R}_G$-class C, the balls of radius r_0 about points of $\widehat{Y} \cap C$ are pairwise disjoint. Moreover, since Y is cocompact, there exists an $r_1 > r_0$ such that the balls of radius r_1 about points of $\widehat{Y} \cap C$ cover all of C. Therefore, each Dirichlet region $D(\widehat{y})$ is the interior of a bounded hyperbolic polygon, and for each $\widehat{R}_G$-class C, we have $\bigcup_{\widehat{y} \in \widehat{Y} \cap C} \overline{D(\widehat{y})} = C$. Then the complement of all of the Dirichlet regions naturally carves out a Borel graph $\mathcal{H}$ on the Borel set $Z \subseteq K\backslash X$, of all points which are vertices of a boundary polygon of one of the Dirichlet regions, and we have $\mathcal{R}_{\mathcal{H}} = (\widehat{R}_G)_{\restriction Z}$. Let $\widehat{\mathcal{G}}$ be the Borel graph on $\widehat{Y}$ obtained as the dual graph to $\mathcal{H}$, i.e., $\widehat{y}_0$ and $\widehat{y}_1$ are connected by an edge e^* if and only if the hyperbolic polygons $\overline{D(\widehat{y}_0)}$ and $\overline{D(\widehat{y}_1)}$ share an edge e. Then $\mathcal{R}_{\widehat{\mathcal{G}}} = (\widehat{R}_G)_{\restriction \widehat{Y}}$. For each $z \in Z$ let $B_z = \{e^* : e \in \mathcal{H} \text{ is incident with } z\}$. Then the collection $\{B_z : z \in Z\}$ is a Borel 2-basis for $\widehat{\mathcal{G}}$, so by Theorem 3.6 there exists a $\widehat{\nu}$-conull $\mathcal{R}_{\widehat{\mathcal{G}}}$-invariant set $\widehat{Y}_0 \subseteq \widehat{Y}$ and an acyclic Borel subgraph $\widehat{\mathcal{G}}_0 \subseteq \widehat{\mathcal{G}}$ with $(\mathcal{R}_{\widehat{\mathcal{G}}_0})_{\restriction \widehat{Y}_0} = (\mathcal{R}_{\widehat{\mathcal{G}}})_{\restriction \widehat{Y}_0} = (\widehat{R}_G)_{\restriction \widehat{Y}_0}$.

Let $Y_0 \subseteq Y$ be the preimage of $\widehat{Y}_0$ under the projection to $K\backslash X$. By our choice of Y, each $\mathcal{R}_K$-class contains at most one point from Y. Then $\widehat{\mathcal{G}}_0$ lifts to a treeing $\mathcal{G}_0$ of $(\mathcal{R}_G)_{\restriction Y_0}$. □

Corollary 4.2. *The groups* $\mathrm{Isom}(\mathbf{H}^2)$, $\mathrm{PSL}_2(\mathbf{R})$, *and* $\mathrm{SL}_2(\mathbf{R})$ *are all measure strongly treeable, as are all of their closed subgroups; thus, for instance, all surfaces groups* $\pi_1(\Sigma_g)$, $g \geq 2$, *are measure strongly treeable. In particular, each of these groups has fixed price.*

Proof. This follows since measure strong treeability is closed under taking closed subgroups, and a compact by measure strongly treeable group is itself measure strongly treeable (Theorem B.3). The last statement follows from Proposition A.11. □

Remark 4.3. To compute the actual cost of these lcsc groups G (with respect to a fixed Haar measure λ), one may use the formula $\mathrm{cost}(G,\lambda) - 1 = (\mathrm{Cost}(\Gamma) - 1)/\mathrm{covol}_\lambda(\Gamma)$, where Γ is any lattice of G.

4.1. Groups with planar Cayley graphs.

Theorem 4.4. *Let Γ be a finitely generated group and suppose that Γ has a Cayley graph which is planar. Then Γ is measure strongly treeable.*

Proof. By [Dro06, Theorem 5.1], Γ is finitely presented. It follows from Dunwoody's accessibility Theorem [Dun85] that Γ is the Bass-Serre fundamental group of a finite graph of groups whose edge groups are all finite and whose vertex groups are finite or one-ended with planar Cayley graphs (by [Bab77]). Therefore, by Theorem B.3-(8), it suffices to prove the theorem when Γ has one end. In this case, it is 2-vertex-connected and by Thomassen's Theorem (Theorem 3.1) it admits an accumulation-free planar embedding. If Γ is amenable then Γ is measure strongly treeable by [CFW81]. Otherwise, by [LS77, Section III.8] Γ is isomorphic to a discrete subgroup of $\mathrm{Isom}(\mathbf{H}^2)$, so we are done by Corollary 4.2.

□

Corollary 4.5. *Surface groups are measure strongly treeable.*

Hjorth's theorem [Hjo06] implies that the orbit equivalence relation of any free p.m.p. action of a surface group (of integer cost) is also generated by a free action of a free group of the same cost. Does the converse hold? That is:

Question 4.6. *Can any orbit equivalence relation of a free p.m.p. action of a free group also be generated by a free action of any surface group of appropriate cost?*

5. Elementarily free groups and towers

A group is said to be **elementarily free** if it satisfies the same first-order sentences as a free group. In [BTW07], it is shown that every finitely generated elementarily free group is treeable. In this section, we show that such groups are even measure strongly treeable.

Theorem 5.1. *Every finitely generated elementarily free group is measure strongly treeable.*

As in [BTW07], our argument relies crucially upon the description of the finitely generated elementarily free groups as fundamental groups of certain "tower" spaces defined inductively (see Theorem 5.2 below) uncovered in [Sel06, Proposition 6] (cf. also [KM98]) and completed later in [GLS20].

We do not intend to state a characterisation of the elementarily free groups and we will content ourselves with the following construction (Theorem 5.2).

Let U be a path-connected CW-complex and let Σ be a connected compact surface with $k \geq 1$ boundary components and with $\pi_1(\Sigma) \simeq \mathbf{F}_d$, with $d \geq 2$ (equivalently its Euler characteristic satisfies $\chi(\Sigma) \leq -1$). Let V be the quotient of $U \sqcup \Sigma$ by identifying each boundary components β_j of Σ with a loop γ_j in U corresponding to infinite order elements of $\pi_1(U)$. We say that the space V is an **IFL**[3] **over** U.

Let $q\colon U \sqcup \Sigma \to V$ be the corresponding quotient map and let q_* denote the induced map on fundamental groups, using a common base point $u \in U$. Then (by standard Bass-Serre theory) q_* realizes an isomorphism $q_*(\pi_1(U)) \simeq \pi_1(U)$.

If we iterate inductively this construction, via a sequence $V_0, V_1, \cdots, V_n$ of CW-complexes where each V_{h+1} is an IFL over V_h, we say that the CW-complex V_n is an **IFL tower over** V_0.

Theorem 5.2 ([Sel06], [KM98],[GLS20]). *Every finitely generated elementarily free group can be described as the fundamental group of an IFL tower over a graph.*

Both theorems are proved in section 5.4.

Remark 5.3. The methods of this section establish measure strong treeability for fundamental groups of a larger class of spaces than just elementarily free groups.

In fact, the characterisation of elementarily free groups requires moreover the existence of a retraction, and more generally of a non-degenerate map for extended towers [GLS20, Definition 4.5, Proposition 4.21], from the fundamental groups of V_{h+1} to that of V_h satisfying certain conditions in restriction to $\pi_1(\Sigma)$. Our methods do not assume anything like that. For us, it is enough that the space V_{h+1} is an IFL over V_h: the boundary loops β_j of Σ are glued to elements of infinite order in $\pi_1(V_h)$ (thus ensuring the injection of the fundamental groups in the natural associated Bass-Serre decomposition) and the Euler characteristic of the compact surface Σ satisfies $\chi(\Sigma) \leq -1$ (compare [Sel06] where $\chi(\Sigma) \leq -2$ or Σ is a punctured torus). This allows us some exceptional surfaces Σ that are explicitly forbidden for ω-residually free tower spaces: the pair of pants (i.e., the sphere minus three disks) and concerning the non-orientable ones, the projective plane minus two disks and the Klein bottle minus one disk.

Moreover our starting space V_0 of height 0 may include such surfaces as the Klein bottle or the non-orientable genus 3 surface, whose fundamental groups are known to be not even ω-residually free. It may also include as V_0 any space with infinite amenable fundamental group.

5.1. **Measure free factors.** Several notions, first introduced in the framework of p.m.p. equivalence relations, admit direct generalizations to the Borel context (possibly with the presence of a measure). This is the case of the notions of **free product decomposition** $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ and **free product decomposition with amalgation** introduced in [Gab00, Déf. IV.9, Déf. IV.6]. This is also the case for the following: a subgroup $\Lambda \leq \Gamma$ is called a **measure free factor** of Γ if for some p.m.p. free action $\Gamma \curvearrowright^a (X, \mu)$, there exists a subrelation $\mathcal{S} \subseteq \mathcal{R}$ such that $\mathcal{R}_a = \mathcal{R}_{a(\Lambda)} * \mathcal{S}$ (μ-a.e.) [Gab05, Def. 3.1]. See also [Alo14] for more results on measure free factors.

Similarly, a subgroup $\Lambda \leq \Gamma$ is called a **measure strong free factor** if for every free Borel action $\Gamma \curvearrowright^a X$ and for every Borel auxiliary probability measure μ on X, there exists a subrelation $\mathcal{S} \subseteq \mathcal{R}$ such that $\mathcal{R}_a = \mathcal{R}_{a(\Lambda)} * \mathcal{S}$ on a Γ-invariant subset

[3] IFL may stand for Injective or Inductive - Free or Fuchsian - Level or Loft.

$X_0 \subseteq X$ of full μ-measure. If moreover the subrelation $\mathcal{S}$ can always be chosen to be treeable, then Λ is called a **measure strong free factor** of Γ **with treeable complement**.

The main technical result of this section is the following.

Theorem 5.4. *If V is an IFL over U, then $\pi_1(U) = q_*(\pi_1(U))$ is a measure strong free factor of $\pi_1(V)$ with treeable complement.*

Corollary 5.5. *If moreover $\pi_1(U)$ is treeable, strongly treeable, or measure strongly treeable, then the same holds for $\pi_1(V)$.*

This corollary follows immediately from Theorem 5.4 (using co-induction when $\pi_1(U)$ is only treeable).

Example 5.6. Take a pair of pants and glue each of its boundary components to a single extra circle with indices $d_1, d_2, d_3 \in \mathbf{Z} \setminus \{0\}$ (and similarly for the two other exceptional allowed surfaces, the projective plane minus two disks or the Klein bottle minus one disk, as soon as $\sum_i |d_i| \geq 3$). The resulting group is not elementarily free by [GLS20, Lemma 10.20] but it is measure strongly treeable by Theorem 5.4.

We also obtain an extension of [Gab05, Th. 3.3] to the non p.m.p. case and for every free actions.

Corollary 5.7. *Let Γ be the free group on $a_1, \ldots, a_n, b_1, \ldots, b_n$. The product of commutators $\prod [a_i, b_i]$ is a measure strong free factor in Γ. The analogous statement holds in the free group on $a_1, \ldots, a_n$ ($n \geq 2$) for the product of the squares: $a_1^2 \cdots a_n^2$.*

It follows that fundamental groups of "branched surfaces" from [Gab05] are measure strongly treeable, and in particular strongly treeable. Similarly to [Gab05, Question 3.10], we ask the question:

Question 5.8. *What are all the measure strong free factors of the free group $\mathbf{F}_r$?*

5.2. **Extended IFL towers - Non connected grounds.** Let Σ be a connected compact surface with $k > 0$ boundary components $\{\beta_j \colon 1 \leq j \leq k\}$ such that $\pi_1(\Sigma)$ is a non-cyclic free group. Consider a collection of disjoint CW-complexes $U_1, U_2, \cdots, U_r$ and a collection $\{\gamma_j \colon 1 \leq j \leq k\}$ of loops of infinite order in $\pi_1(U_{f(j)})$ for some surjective map $f : \{1 \leq j \leq k\} \to \{1 \leq i \leq r\}$. Let W be the space obtained by glueing Σ to the $\sqcup U_i$ by attaching its j-th boundary component to γ_j. This connected space W is an **extended IFL over** the U_i's.
Let U^* be the connected space obtained by attaching for each $i = 2, 3, \cdots, r$ an arc ξ_i from (the base point of) U_1 to (the base point of) U_i.
Let V be the space obtained from W by adding the arcs ξ_i.
On the one hand, Theorem 5.4 ensures that $\pi_1(U^*) = \pi_1(U_1) * \pi_1(U_2) * \cdots * \pi_1(U_r)$ is a measure strong free factor of $\pi_1(V)$ with treeable complement and, on the other hand, $\pi_1(V) = \pi_1(W) * \mathbf{F}_{r-1}$. Iterating the construction would of course produce **extended IFL towers over** the first space.

Corollary 5.9. *Let W be an extended over $U_1, U_2, \cdots, U_r$. If the $\pi_1(U_i)$ are all treeable, strongly treeable, or measure strongly treeable, then the same holds for $\pi_1(W)$.*

Remark 5.10. Observe that there is not too much to expect in terms of measure strong free factor involving all of the $\pi_1(U_i)$, as examplified by Corollary 5.11.

Proof of Corollary 5.9. When the $\pi_1(U_i)$ all satisfy one of the properties, then their free product also does (Theorem B.3-(8)). By Corollary 5.5, the same then holds for $\pi_1(V)$. It follows from Theorem B.3-(5) that treeability, strong treeability, and measure strong treeability respectively pass from $\pi_1(V)$ to its subgroup $\pi_1(W)$. $\square$

Corollary 5.11. *Let $r \geq 3$ and $\Gamma_1, \Gamma_2, \cdots, \Gamma_r$ be countable groups and let $\gamma_i \in \Gamma_i$ be an infinite order element for each $i = 1, 2, \cdots, r$. If the Γ_i are treeable, strongly treeable, or measure strongly treeable, then the same holds for*

$$\Gamma = \left\langle \Gamma_1, \Gamma_2, \cdots, \Gamma_r \,\middle|\, \prod_{i=1}^{r} \gamma_i = 1 \right\rangle = (\Gamma_1 * \Gamma_2 * \cdots * \Gamma_r) / \langle\langle \prod_{i=1}^{r} \gamma_i \rangle\rangle.$$

As a particular case, we recover $\mathfrak{MST}$ for the group $\Gamma = \langle a_1, a_2, s \,|\, a_1^2 a_2^2 s^d = 1\rangle$ of Example 5.6 with the punctured Klein bottle.

Remark 5.12. Observe that the analogous statement fails for $r = 2$ as witnessed by the group $\Gamma = \langle t_1, t_2 \,|\, t_1^{d_1} t_2^{d_2} = 1\rangle$ which has cost 1 (as an amalgamated product $\mathbf{Z} *_{\mathbf{Z}} \mathbf{Z}$) and is non-treeable when non-amenable [Gab00, Corollary VI.22], e.g., when the indices $|d_i|$ are ≥ 3.

Proof. This is exactly the fundamental group of an extended IFL over spaces U_i with $\pi_1(U_i) \simeq \Gamma_i$ when Σ is a sphere minus r disks. $\square$

5.3. **Proof of Theorem 5.4.**

Proof of Theorem 5.4. Fix an enumeration $\{\beta_j \colon 1 \leq j \leq k\}$ of the (disjoint) boundary circles of Σ based at $u_j \in \beta_j$. Fix also for all $j \geq 2$ (if there are any) simple paths t_j from u_j to u_1 whose interiors are mutually disjoint and disjoint from the β_j. For sake of notation, set t_1 to be the trivial path at u_1. Let b_j be the corresponding loops based at u_1 (and their class in $\pi_1(\Sigma, u_1)$), i.e., $b_j = t_j^{-1} \beta_j t_j$ (for $j = 1, \cdots, k$). Let $\{a_i \colon i \leq l\}$ enumerate the other generators of a standard presentation of $\pi_1(\Sigma, u_1)$ as a surface with boundary with a single relation $R(a_i, b_j)$. More precisely:

- If Σ is orientable of genus g then $\chi(\Sigma) = -2g - k + 2$ and the standard presentation is given by $\pi_1(\Sigma) = \langle \{a_i\}_{i=1}^{2g}, \{b_j\}_{j=1}^{k} \,|\, R(a_i, b_j)\rangle$ with $R(a_i, b_j) = \prod_{i=1}^{g} [a_{2i-1}, a_{2i}] \prod_{j=1}^{k} b_j$. We set $l = 2g$.
- If Σ is non-orientable of genus g then $\chi(\Sigma) = -g - k + 2$ and the standard presentation is given by $\pi_1(\Sigma) = \langle \{a_i\}_{i=1}^{g}, \{b_j\}_{j=1}^{k} \,|\, R(a_i, b_j)\rangle$ with $R(a_i, b_j) = \prod_{i=1}^{g} a_i^2 \prod_{j=1}^{k} b_j$. We set and $l = g$.

Notice that $\pi_1(\Sigma)$ is isomorphic to a free group on $l + k - 1$ generators since $k \geq 1$.

Claim. *The group $\pi_1(V, u)$ splits as a graph-of-groups on two vertices, with vertex groups $\pi_1(U, u)$ and $\pi_1(\Sigma, u)$, and with k edges joining the two vertices with edge groups isomorphic to $\langle b_j \rangle$.*

$$\pi_1(V, u) = \langle \pi_1(U, u), a_i, t_j \,|\, R(a_i, t_j^{-1} \gamma_j t_j) = t_1 = 1\rangle.$$

Remark 5.13. Conversely, a group admitting such a graph-of-groups splitting with γ_j of infinite order in $\pi_1(U, u)$ satisfies the assumptions of Theorem 5.4.

Proof of the claim. The claim is quite clear, but we take advantage of this part to introduce our notations. Let α_i be simple loops based at u_1 representing a_i which are pairwise disjoint and disjoint from each t_j and b_j (see Figure 2).

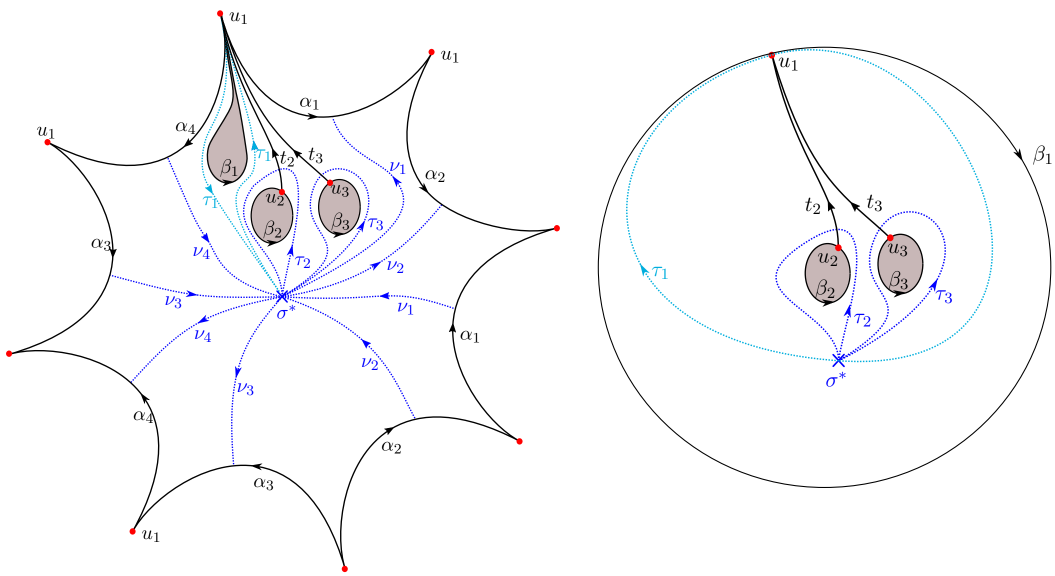

FIGURE 2. The graphs $\mathtt{G}_1$ and $\mathtt{G}^*$ for a genus two surface minus three disks and for a sphere minus three disks

The vertices $\{u_j\}$ and the paths $\{\alpha_i, \beta_j, t_j\}$ draw a connected graph $\mathtt{G}_1$ on Σ. Its complement is a connected 2-cell σ, homotopically equivalent to a disk. This defines a cellular decomposition $\mathcal{C}$ of Σ.

The polyhedral dual graph $\mathtt{G}^*$ has one vertex σ^* (in the interior of σ). Its edges have two forms: τ_j is a loop turning around β_j and crossing t_j, while ν_i is a loop crossing α_i. Observe that they do not cross the β_j. The loop τ_1 has a particular status since $t_1 \equiv u_1$ being trivial, τ_1 contains u_1. Removing τ_1, the family $\nu_1, \nu_2, \cdots, \nu_l, \tau_2, \cdots, \tau_k$ freely generates the fundamental group $\pi_1(\Sigma, \sigma^*)$.

Up to an homotopy equivalence, we may glue each boundary component β_j to the loop γ_j based at the base point u of U in such a way that u_j is glued to u. Then all the u_j become identified with u in V and each t_j (for $j = 2, \cdots, k$) becomes a loop. Performing this identification in Σ produces the graph $\mathtt{G}$ obtained from $\mathtt{G}_1$ by identifying the u_j all together. The graph $\mathtt{G}^*$ remains unchanged.

Turning to the fundamental group based at u and seeing the quotient space $V = (U \sqcup \Sigma)/_{\{\beta_j = \gamma_j\}}$ as a graph-of-spaces obtained by first gluing β_1 (furnishing an amalgam) and then in succession the other β_j (iteratively furnishing HNN-extensions with free letter t_j), it follows that $\pi_1(V,u)$ is described as a graph-of-groups on two vertices, with vertex groups $\pi_1(U,u)$ and $\pi_1(\Sigma,u)$, and with k edges joining the two vertices with edge groups isomorphic to $\langle b_j\rangle$.

By assumption, each $q_*(\gamma_j)=q_*(\beta_j)$ has infinite order in $\pi_1(U,u)$, hence the edge groups inject into the vertex groups and thus both vertex groups inject into $\pi_1(V,u)$ by [Ser77, Theorem 11, Corollary 1]. Then $\pi_1(V)$ is generated above $q_*\pi_1(U)$ by adding $\{a_i\colon i\leq l\}$ and $\{t_j\colon 2\leq j\leq k\}$ with a unique relation $R(a_i,t_j^{-1}\gamma_j t_j)$. $\square$

We now discuss the relevant geometry of the Cayley graph of $\pi_1(V)$ obtained by adding generators a_i,t_j as above to $q_*\pi_1(U)$, with the goal of eventually applying this reasoning measurably across orbits of an action of $\pi_1(V)$ on a standard probability space to prove the proposition.

On each connected component of the pull back of U and Σ, under the universal covering $p:\widetilde{V}\to V$, the map p coincides with a regular covering $p_U:\widehat{U}\to U$ and $p_\Sigma:\widehat{\Sigma}\to\Sigma$ respectively. Indeed, by injection of the π_1's and of the $\langle t_j\rangle$ $(2\leq j\leq k)$ in $\pi_1(V)$, the coverings p_U and p_Σ are universal coverings and $p^{-1}(U)$ and $p^{-1}(\Sigma)$ is precisely a disjoint union of copies of $\widehat{U}$ and $\widehat{\Sigma}$ respectively.

Choose a base point $\tilde{u}$ in $\widetilde{V}$ above u. These copies form the vertex spaces of the bipartite tree-of-spaces decomposition of $\widetilde{V}$ (mimicking the Bass-Serre tree of the graph-of-groups).

Again by injectivity, the boundary components β_j pull-back to disjoint families of bi-infinite paths in $p^{-1}(\Sigma)\subseteq\tilde{V}$. Observe that the pull-back graph $\widehat{\mathtt{G}}=p_\Sigma^{-1}(\mathtt{G})$ in $\widehat{\Sigma}$ connects the various boundary components $p_\Sigma^{-1}(\{\beta_1,\cdots,\beta_k\})$, but in a non unique way as witnessed by the relation $R(a_i,t_j^{-1}\gamma_j t_j)$, i.e., by the boundary cycle of each of the pull-backs of the 2-cell σ under p_Σ. As in previous arguments, selecting a one-ended spanning subforest $\mathtt{T}^*$ of the dual graph $\widehat{\mathtt{G}}^*=p_\Sigma^{-1}(\mathtt{G}^*)$ and removing all the edges of $\widehat{\mathtt{G}}$ that cross an edge of $\mathtt{T}^*$ resolves this non-uniqueness issue since the resulting subgraph $\mathtt{T}$ of $\widehat{\mathtt{G}}$ remains connected but becomes acyclic. It contains the boundary lines $p_\Sigma^{-1}(\{\beta_1,\cdots,\beta_k\})$. What we are really after is the complement

$$\mathtt{T}_0:=\mathtt{T}\setminus p_\Sigma^{-1}(\{\beta_1,\cdots,\beta_k\}),$$

since this is the part of $\mathtt{T}$ that lies outside of $p^{-1}(U)$.

Observe that even though some boundary components β_i,β_j may be identified in V for instance when $\gamma_i=\gamma_j$, their pull-backs in $\widetilde{V}$ are different (again by injectivity of $\pi_1(\Sigma)$ in $\pi_1(V)$ and $\widehat{\Sigma}=\widetilde{\Sigma}$).

Selecting a one-ended spanning subforest in $p^{-1}(\mathtt{G}^*)$ (that is, in each copy of $p_\Sigma^{-1}(\mathtt{G}^*)$) and removing the edges it crosses as above in each connected component $\widehat{\Sigma}$ of $p^{-1}(\Sigma)$ leads to a subspace $\widetilde{V}^t$ of $\widetilde{V}$ where each vertex space of type $\widehat{\Sigma}$ is now replaced by a tree connecting its boundary components.

We claim that "$\widetilde{V}^t$ is a tree above the $p^{-1}(U)$." More precisely, contracting each component of type $\widehat{U}$ to a point produces a graph Ψ which we claim is a tree. Indeed, if θ is a cycle without backtracking in the resulting graph Ψ, we pull it back to the tree-of-spaces $\widetilde{V}^t$ and select there a backtracking point (entering a vertex space and leaving it by the same edge space). This cannot happen in a vertex space of type $\widehat{U}$ since θ has no backtracking when this space is contracted to a point. This cannot happen in a vertex space of type $\widehat{\Sigma}$ since these are now forests. It follows that θ is trivial.

We collect some useful observations about the dual graph $\widehat{\mathtt{G}}^* = p_\Sigma^{-1}(\mathtt{G}^*)$:

(1) $\widehat{\mathtt{G}}^*$ is dual to the pull-back $\widehat{\mathcal{C}} = p_\Sigma^{-1}(\mathcal{C})$ of the cellulation $\mathcal{C}$ of Σ,
(2) $\widehat{\mathtt{G}}^*$ is planar (like $\widehat{\Sigma}$),
(3) $\widehat{\mathtt{G}}^*$ is connected (since $\mathtt{G}^*$ in Σ generates the fundamental group of Σ), and
(4) $\widehat{\mathtt{G}}^*$ is quasi-isometric with the covering group $\pi_1(\Sigma)$ of $\widehat{\Sigma}$. In particular, it is not 2-ended.

Finally, towards proving Theorem 5.4, suppose that $\Gamma = \pi_1(V)$ acts in a free, Borel fashion on the standard probability space (X, μ). We will find a μ-a.e. acyclic subgraph $\mathcal{T}_0$ of the Borel graph given by the a_i, t_j such that $\mathcal{R}_{\pi_1(V)} = \mathcal{R}_{q_*(\pi_1(U))} * \mathcal{T}_0$. Consider the quotient by the diagonal action $\mathcal{V} = \Gamma\backslash(X\times\widetilde{V})$. Using the isomorphism $X \simeq X\times\{\tilde{u}\}$, we obtain on X the Borel graph $\mathcal{G} = \Gamma\backslash(X\times p^{-1}(\mathtt{G}))$ which corresponds to the graphing Φ given by the action of the elements $\{a_i\}_{i\in[1..l]}, \{b_j\}_{j\in[1..k]}, \{t_j\}_{j\in[2..k]}$ of Γ. The dual graph $p^{-1}(\mathtt{G}^*) \subseteq \widetilde{V}$ delivers a Borel graph $\mathcal{G}^*$ with the natural Borel probability measure μ^* on its vertices $X^* = \Gamma\backslash(X \times p^{-1}(\sigma^*))$. The pull-back from $\tilde{u}$ in $\widetilde{V}$ of a simple path in $\sigma \subseteq \Sigma \subseteq V$ from u to σ^* entails a Borel isomorphism $X \xrightarrow{\sim} X^*$ used to push forward the measure μ to μ^*. By condition (4) above $\mathcal{G}^*$ is nowhere 2-ended and it comes with a Borel 2-basis by (2). In fact $\mathtt{G}^*$ is a tree, thus Theorem 2.3 could be applied instead. By Corollary 3.7, $\mathcal{G}^*$ has a Borel μ^*-a.e. one-ended spanning subforest $\mathcal{T}^*$.

Applying the by now standard procedure above of removing from $\mathcal{G}$ the edges that cross $\mathcal{T}^*$ and those associated with the generators $\{b_j\}_{j\in[1..k]}$ leads to an acyclic Borel graph $\mathcal{T}_0$ above the graph given by the restriction of the action $\Gamma \curvearrowright X$ to the subgroup $q_*(\pi_1(U)) \leq \Gamma$, giving

$$\mathcal{R}_{\pi_1(V)} = \mathcal{R}_{q_*(\pi_1(U))} * \mathcal{T}_0$$

μ-almost everywhere. This completes the proof of Theorem 5.4. □

Remark 5.14. Observe in the proof of Theorem 5.4 that the treeing $\mathcal{T}_0$ eventually created is obtained by restricting the graphing given by the elements $a_i, t_j \in \pi_1(V, u) = \langle \pi_1(U, u), a_i, t_j \mid R(a_i, t_j^{-1}\gamma_j t_j) = t_1 = 1\rangle$.

5.4. Back to elementarily free groups.

Proof of Theorem 5.2. This Theorem follows mainly from the description (by hyperbolic ω-residually free tower spaces) in [Sel06, Proposition 6], except that in order to include all the elementarily free groups, one also needs to consider extended towers (as explained in [GLS20, Section 10.3, Definition 4.26, Corollary 7.10.5]), i.e., a generalization of the IFL construction[4] V where U is not necessarily connected: The boundary components of Σ are attached to (infinite order loops in) the connected components $U_1, U_2, \cdots, U_s$ of U (see also Section 5.2).

Indeed, when $s \geq 2$, we can "force" U to be connected by decomposing $\Sigma = \Sigma' \# S$ as a connected sum where S is a sphere minus $s+1$ disks. The space V has a similar decomposition where we first attach one boundary component of S to each U_i and then attach Σ' via the connected sum to S and along its remaining boundary components to the U_i so as to recover our generalized IFL construction (see [GLS20, Proposition 5.1 and its pictures]). Thus attaching S amounts to connect the U_i's with the result of taking the free product of the $\pi_1(U_i)$. The only problem would be if Σ' would not satisfy the condition that $\chi(\Sigma') \leq -1$ (compare [GLS20, Remark 10.3] where Σ' must be non-exceptional). This problem is ruled out for us by [GLS20, Remark 4.25] which claims that (for extended towers producing elementarily free groups one can assume that $g(\Sigma) \geq n_1$, where n_p is the number of the U_i's to which p boundary components of Σ are attached and where g denotes the genus. Denoting by k the number of boundary components, we have $k(\Sigma) = \sum_p p n_p$. Concerning Σ', we have $g(\Sigma') = g(\Sigma) \geq n_1$ and $k(\Sigma') = k(\Sigma) + 1 - \sum_p n_p = 1 + \sum_p (p-1) n_p \geq 1$. It follows that $g(\Sigma') + k(\Sigma') \geq 3$ and $\chi(\Sigma') \leq -1$. □

Proof of Theorem 5.1. Thanks to Theorem 5.2, simply iterate Theorem 5.4 through the levels of an IFL tower with desired fundamental group, after applying Theorem 4.4 to the level 0 group. □

Remark 5.15. Observe that we could have used another characterization of the finitely generated elementarily free groups from [GLS20, Corollary 7.10.6 or 7.10.7] as those groups Γ for which there exists a free group $\mathbf{F}$ such that $\Gamma * \mathbf{F}$ is the fundamental groups of a simple (= non-extended) hyperbolic ω-residually free tower space in the sense of Sela [Sel06] with ground spaces V_0 given by compact connected graphs with non-zero Euler characteristics. Then applying Theorem 5.4 for such a simple tower and Theorem B.3-(5) to deduce the treeability properties of Γ from those of $\Gamma * \mathbf{F}$ would have delivered an analternative proof of Theorem 5.1. However, the approach using Theorem 5.2 with Theorem 5.4 gives a little bit more: it gives the additional structure of successive measure strong free factors with treeable complements. Compare Section 5.2 and Remark 5.10.

As in Question 4.6, we ask whether a version of Hjorth's theorem [Hjo06] holds for elementarily free groups:

Question 5.16. *Given a free group and an elementarily free group with the same cost, can every orbit equivalence relation of a free p.m.p. action of the former also be generated by a free action of the latter?*

[4] Technically, there is another type of level (étage) in [GLS20] consisting in taking free products of $\pi_1(V_j)$ with $\mathbf{Z}$, but this kind of level can be moved down to V_0.

6. Ergodic dimension of aspherical n-manifolds

The **ergodic dimension of a group** Γ [Gab02a, Déf. 6.4] is the smallest geometric dimension of the orbit-equivalence relations $\mathcal{R}_\alpha$ among all of its free p.m.p. actions $\Gamma \curvearrowright^{\alpha} (X,\mu)$ (see [Gab21] for more on this notion). The goal of this section is to prove the following theorems:

Theorem 6.1. *Suppose Γ is the fundamental group of a compact aspherical connected manifold M (possibly with boundary) of dimension at least 2. Then all free p.m.p. actions of Γ have ergodic dimension at most $\dim(M)-1$.*

Recall that M is **aspherical** if its universal cover $\tilde{M}$ is contractible. A manifold is **closed** if it is compact without boundary.

Theorem 6.2. *Suppose Γ is the fundamental group of a closed aspherical connected manifold of dimension 3. Then either*

(1) Γ is amenable, or
(2) Γ has strong ergodic dimension 2.

6.1. Removing a dual one-ended subforest.

We start by considering the effect of removing from a contractible complex the cells associated with a one-ended subforest of its dual complex.

Let M be a smooth compact connected manifold of dimension d, possibly with boundary. Let $\bar{\Sigma}$ be a triangulation of M. It induces a triangulation of its boundary ∂M. Pick a base point u among the vertices of $\bar{\Sigma}$. The induced triangulation Σ of the universal cover $\tilde{M}$ is invariant under the covering group $\pi_1(M,u)$, once a pull-back $\tilde{u}$ of u is chosen.

A $(d-1)$-dimensional cell is called **singular** if it belongs to the boundary of $\tilde{M}$, i.e., if it is a face of a single d-dimensional cell. A $(d-1)$-dimensional cell is called **regular** otherwise, i.e., if it is a face of exactly two d-dimensional cells.

We define the **dual graph** $\mathtt{G}^*$ to Σ. Its set of vertices $\mathtt{V}(\mathtt{G}^*)$ are in bijection with the set $\Sigma^{(d)}$ of d-dimensional cells of Σ. We denote the natural bijection by

$$D_d : \mathtt{V}(\mathtt{G}^*) \to \Sigma^{(d)}.$$

Its set of edges $\mathtt{E}(\mathtt{G}^*)$ is in bijection with the set $\Sigma_r^{(d-1)}$ of regular $(d-1)$-dimensional cells of Σ via

$$D_{d-1} : \mathtt{E}(\mathtt{G}^*) \to \Sigma_r^{(d-1)}$$

where an edge e joins two vertices v_1, v_2 if and only if $D_{d-1}(e)$ is a common face of the d-dimensional cells $D_d(s_1), D_d(s_2)$.

Let $\mathtt{F}^*$ be a one-ended spanning subforest of the dual graph $\mathtt{G}^*$ of Σ. Using the same notation as in Section 3.1, we define the simplicial subcomplex $\Sigma \overset{*}{\ominus} \mathtt{F}^*$ of Σ by removing all the d-dimensional cells from Σ and all the $(d-1)$-dimensional cells corresponding to an edge in $\mathtt{F}^*$, i.e.,

$$\Sigma \overset{*}{\ominus} \mathtt{F}^* := \Sigma \setminus \left(D_d(\mathtt{V}(\mathtt{F}^*)) \cup D_{d-1}(\mathtt{E}(\mathtt{F}^*)) \right).$$

Observe that Σ and $\Sigma \overset{*}{\ominus} \mathtt{F}^*$ have the same k-skeleton for $k \leq d-2$.

Proposition 6.3. *If Σ is contractible then the $(d-1)$-dimensional complex $\Sigma \circledast \mathtt{F}^*$ is contractible.*

Proof of Proposition 6.3. By Whitehead theorem [Whi49], it is enough to show that for all integers $k \geq 0$, the $\pi_k(\Sigma \circledast \mathtt{F}^*, x)$ are trivial.

Let $I^k := [0,1]^k$ be the k-dimensional cube and ∂I^k its boundary.

Consider a continuous map $f : (I^k, \partial I^k) \to (\Sigma \circledast \mathtt{F}^*, x)$ (sending ∂I^k to x) and consider its composition $j \circ f$ with the inclusion $j : \Sigma \circledast \mathtt{F}^* \hookrightarrow \Sigma$. Since Σ has trivial π_k, there is a homotopy $\tilde{H} : (I^k \times I, (\partial I^k) \times I) \to (\Sigma, x)$ between $j \circ f(\cdot) = \tilde{H}(\cdot, 1)$ and the constant map $\tilde{H}(\cdot, 0) : I^k \times \{0\} \to \{x\}$.

We define the compact K_0 as the smallest subcomplex of Σ that contains the image $\tilde{H}(I^k \times I)$. We will show that K_0 is contained in a compact subcomplex $K \subseteq \Sigma$ which admits a retraction $U : K \to K \cap (\Sigma \circledast \mathtt{F}^*)$, i.e., a continuous map whose restriction to $K \cap (\Sigma \circledast \mathtt{F}^*)$ is the identity. The composition $U \circ \tilde{H} : (I^k \times I, (\partial I^k) \times I) \to (\Sigma \circledast \mathtt{F}^*, x)$ will provide a homotopy in $(\Sigma \circledast \mathtt{F}^*, x)$ connecting f to the trivial map $I^k \to \{x\}$, thus proving that $\pi_k(\Sigma \circledast \mathtt{F}^*, x)$ is trivial.

We think of the one-ended forest $\mathtt{F}^*$ as an oriented graph with the orientation pointing toward the single infinite end in each connected component. Every vertex $v \in \mathtt{V}(\mathtt{F}^*) = \mathtt{V}(\mathtt{G}^*)$ has thus exactly one outgoing edge; let's call it $o(v)$. The $\mathtt{F}^*$-back-orbit of v is the unique finite component of $\mathtt{F}^* \setminus o(v)$. It contains v. An edge $e \in \mathtt{E}(\mathtt{F}^*)$ is the out-going edge of a single vertex $\iota(e)$; the $\mathtt{F}^*$-back-orbit of e is the $\mathtt{F}^*$-back-orbit of $\iota(e)$. If e is an edge in $\mathtt{E}(\mathtt{G}^*) \setminus \mathtt{E}(\mathtt{F}^*)$, its $\mathtt{F}^*$-back-orbit is set to be empty. The **$\mathtt{F}^*$-back-orbit saturation** of a finite set $A \subseteq \mathtt{V}(\mathtt{F}^*) \cup \mathtt{E}(\mathtt{F}^*)$ is the union of A with the $\mathtt{F}^*$-back-orbit of all its elements. It is also finite. A finite set is **$\mathtt{F}^*$-back-orbit saturated** if it coincides with its $\mathtt{F}^*$-back-orbit saturation

We transfer these notions to the subcomplexes of Σ via the maps D_d, D_{d-1}. The $\mathtt{F}^*$-back-orbit of a $(d-1)$-cell τ of Σ is the image under D_d and D_{d-1} of the $\mathtt{F}^*$-back-orbit saturation of $D_{d-1}^{-1}(\tau)$. The **$\mathtt{F}^*$-back-orbit saturation** of a finite subcomplex $K_0 \subseteq \Sigma$ is the union of K_0 with the $\mathtt{F}^*$-back-orbit of all its $(d-1)$-cells. A finite subcomplex $K \subseteq \Sigma$ is **$\mathtt{F}^*$-back-orbit saturated** if it coincides with its $\mathtt{F}^*$-back-orbit saturation.

Lemma 6.4. *If L is a compact $\mathtt{F}^*$-back-orbit saturated subcomplex of Σ, then it admits a retraction to $L \cap (\Sigma \circledast \mathtt{F}^*)$.*

Proof. The key point is that any simplex admits a retraction to its boundary minus any one of its faces. We then argue by a decreasing induction on the number of d-dimensional cells of L:

Observe if L has no d-dimensional cells then it has to be contained in $\Sigma \circledast \mathtt{F}^*$.

Otherwise, pick ϵ one of the $n \geq 1$ cells of dimension d of L. Consider in $\mathtt{F}^*$ the unique path p from $D_d^{-1}(\epsilon)$ to the infinite end. By finiteness of L there is a least vertex $v \in p \cap D_d^{-1}(L)$. By definition $\sigma := D_d(v)$ and $\tau := D_{d-1}(o(v))$ are cells of L.

Claim: Removing σ and τ from L delivers a $\mathtt{F}^*$-back-orbit saturated subcomplex $L_\sigma \subseteq L$ of Σ with one d-dimensional cell less, together with a retraction $U_\sigma : L \to L_\sigma$:

by construction, the cell σ doesn't belong to the back-orbit of any other $(d-1)$-dimensional cell of L than τ. The cell σ admits a retraction V_σ to $(\partial\sigma) \setminus \tau$ (its boundary minus τ), which extends (by the identity) on the rest of L.

Now, a decreasing induction gives successive retractions $U_{\sigma_1}, U_{\sigma_2}, \cdots, U_{\sigma_n}$ whose composition is a retraction from L to a subcomplex contained in $\Sigma \circledast \mathtt{F}^*$.

This completes the proof of the lemma. Applying it to the $\mathtt{F}^*$-back-orbit saturation K of K_0, this completes the proof of the proposition. □

□

6.2. **Growth condition.**

Lemma 6.5. *If M is a compact connected smooth d-dimensional manifold (possibly with boundary) and if Σ is a triangulation of its universal cover $\tilde{M}$ that is the pull-back of a (finite) triangulation of M, then the dual graph $\mathtt{G}^*$ of Σ is quasi-isometric with the fundamental group of M.*

Proof. Indeed, $\mathtt{G}^*$ (as well as Σ) is equipped with a co-compact free action of the $\pi_1(M)$. It is enough to check that $\mathtt{G}^*$ is connected. Given any two vertices of $\mathtt{G}^*$, they correspond to two d-dimensional cells σ and τ of Σ. Any path in $\tilde{M}$ joining two points z_σ and z_τ of their interior (the interior of a connected manifold with boundary is path-connected) can be deformed (with fixed extremities) to a path in $\tilde{M} \setminus \partial\tilde{M}$ that avoids the cells of dimension $\leq d-2$ and that crosses the $(d-1)$-dimensional regular cells in finitely many points). Such a path delivers a path in $\mathtt{G}^*$ joining the two vertices. □

6.3. **Proof of Theorems 6.2 and 6.1.** Let Γ be the fundamental group of a compact aspherical smooth manifold M of dimension $d \geq 2$, possibly with boundary, and let $\Gamma \curvearrowright^a (X, \mu)$ a free p.m.p. action.

Consider a triangulation Σ of its universal cover $\tilde{M}$ which is the pull-back of a (finite) triangulation of M and let $\mathtt{G}^*$ be its dual graph. When equipped with the diagonal action, $\widetilde{\Sigma} := X \times \Sigma$ gives a contractible field of complexes on which the equivalence relation $\mathcal{R}_a$ acts smoothly. Similarly, Γ and $\mathcal{R}_a$ act smoothly on the dual field of graphs $X \times \mathtt{G}^*$. The quotient $\mathcal{G} := \Gamma \backslash (X \times \mathtt{G}^*)$ gives a measure preserving Borel graph on the finite measure space

$$\Gamma \backslash ((X, \mu) \times \mathtt{V}(\mathtt{G}^*)) \simeq \Gamma \backslash \left((X, \mu) \times \Sigma^{(d)} \right) \simeq (X, \mu) \times \Gamma \backslash \Sigma^{(d)}.$$

Since the actions are free almost every connected component of $\mathcal{G}$ is isomorphic with $\mathtt{G}^*$.

If $\mathtt{G}^*$ is finite, so is Γ and it has strong ergodic dimension $= 0$. If $\mathtt{G}^*$ is two-ended, then Γ is two-ended as well (by Lemma 6.5) and thus amenable, and we are done by Ornstein-Weiss Theorem [OW80]: Γ has strong ergodic dimension $= 1$.

Otherwise, Th. 2.1 ensures the existence of an a.e. one-ended subforest $\mathcal{F}$ of $\mathcal{G}$.

Pulling back $\mathcal{F}$ under the quotient map $X \times \mathtt{G}^* \longrightarrow \mathcal{G}$ delivers a smooth $\mathcal{R}_a$-invariant field $\mathcal{F}^*$ of one-ended subforests of $\mathcal{G}^* := X \times \mathtt{G}^*$.

Then consider the smooth $\mathcal{R}_a$-invariant field of simplicial subcomplexes $\widetilde{\Sigma} \circledast \mathcal{F}^*$ obtained by removing from $\widetilde{\Sigma}$ all the d-dimensional cells and all the $(d-1)$-dimensional cells associated with an edge from the forest $\mathcal{F}^*$. This construction is made fiber-wise and the bijections D_d and D_{d-1} extend naturally to the field of complexes framework. The resulting complex $(\widetilde{\Sigma} \circledast \mathcal{F}^*)_x$ above a.e. $x \in X$ is precisely the complex $\widetilde{\Sigma}_x \circledast \mathcal{F}^*{}_x$, of the kind ($\Sigma$ minus cells dual to a one-ended subforest) that has been considered in section 6.1. Applying Prop. 6.3 shows that a.e. complex $(\widetilde{\Sigma} \circledast \mathcal{F}^*)_x$ is contractible (and of course $(d-1)$-dimensional. We have proved Theorem 6.1. $\square$

As for Theorem 6.2 where M is 3-dimensional, we already know by Theorem 6.1 that Γ has ergodic dimension ≤ 2.
In case Γ has ergodic dimension $= 1$, then (by [Gab02a, Cor. 3.17]) the 2nd ℓ^2-Betti number $\beta_2^{(2)}(\Gamma) = 0$. By Poincaré duality [CG86, Sect. 5] we deduce $\beta_1(\Gamma) = 0$. In case M is non-orientable, the orientation cover $\bar{M} \to M$ has fundamental group $\bar{\Gamma}$, an index 2 subgroup of Γ, and we get $2\beta_1^{(2)}(\Gamma) = \beta_1^{(2)}(\bar{\Gamma}) = \beta_2^{(2)}(\bar{\Gamma}) = 2\beta_2^{(2)}(\Gamma) = 0$. By [Gab02a, Prop. 6.10], a group of ergodic dimension $= 1$ with $\beta_1^{(2)} = 0$ is amenable. In case Γ has ergodic dimension $= 0$, then Γ is finite, thus amenable. Thus, if Γ admits a p.m.p. free action of geometric dimension < 2, then it is amenable. Theorem 6.2 is proved. $\square$

A. Treeability for locally compact groups

Definition A.1. Let E be a Borel equivalence relation on a standard Borel space X. We say that E is **Borel treeable** if there is an acyclic Borel graph $\mathcal{T}$ on X with $E_{\mathcal{T}} = E_G$. Given a Borel probability measure μ on X, we say that E is **μ-treeable** if there is a μ-conull Borel subset X_0 of X such that the restriction $E_{\restriction X_0}$ is Borel treeable. We say that E is **measure treeable** if E is μ-treeable for every Borel probability measure μ on X.

Let G be a locally compact second countable (lcsc) group, and let $G \curvearrowright X$ be a free Borel action of G on a standard Borel space X. A subset $Y \subseteq X$ is called a **cross section** of the action $G \curvearrowright X$ if Y is a complete section for the action and if there exists a neighborhood U of 1_G such that the map $U \times Y \to X$, $(g, y) \mapsto g \cdot y$, is injective.

Theorem A.2 ([Kec92])**.** *Let $G \curvearrowright X$ be a Borel action of a lcsc group G on a standard Borel space X. Then there exists a Borel cross section for the action.*

A cross section $Y \subseteq X$ is called **cocompact** if there exists a compact subset $K \subseteq G$ such that $K \cdot Y = X$.

Theorem A.3 (See [Slu17, §2])**.** *Let $G \curvearrowright X$ be a free Borel action of a lcsc group G on a standard Borel space X. Then there exists a Borel cross section for the action which is cocompact. Moreover, given any compact set $L \subseteq G$, we may find a cocompact Borel cross section $Y \subseteq X$ such that $L \cdot y_0 \cap L \cdot y_1 = \varnothing$ for distinct $y_0, y_1 \in Y$.*

The analogous results was known before in the measure-theoretic setting [For74].

Proposition A.4. *Let $G \curvearrowright X$ be a free Borel action of a lcsc group G on a standard Borel space X and let μ be a Borel probability measure on X. Then there exists a quasi-invariant Borel probability measure μ' such that μ and μ' have the same G-invariant null sets.*

Proof. Let m be a probability measure on G which is equivalent to Haar measure. Define $\mu' = m * \mu$, i.e., $\int_X f \, d\mu' = \int_G \int_X g \cdot f \, d\mu \, dm$. □

Definition A.5. Let $G \curvearrowright X$ be a free Borel action of a lcsc group G on a standard Borel space X.

(1) The action is called **Borel treeable** if the equivalence relation $\mathcal{R}_G$ is Borel treeable.
(2) Let μ be a Borel probability measure on X. The action is called **μ-treeable** if there exists a G-invariant μ-conull Borel set $X_0 \subseteq X$ such that $G \curvearrowright X_0$ is Borel treeable.
(3) The action is called **measure treeable** if it is μ-treeable for every Borel probability measure μ.

Proposition A.6. *Let $G \curvearrowright X$ be a Borel action of a lcsc group G on a standard Borel space X. Then the following are equivalence:*

(1) The equivalence relation $\mathcal{R}_G$ is Borel treeable.
(2) There exists a Borel cross section Y such that $(\mathcal{R}_G)_{\restriction Y}$ is Borel treeable.
(3) For every Borel cross section Y the restriction $(\mathcal{R}_G)_{\restriction Y}$ is Borel treeable.

Proof. (1)⇒(3): Let $\mathcal{T}$ be a Borel treeing of $\mathcal{R}_{\mathcal{T}}$ and let Y be a Borel cross section for the action. For each $x \in X$ the set $[x]_{\mathcal{R}_G} \cap Y$ is countable, so the set of points along a path from x to Y of minimal length is also countable. Therefore, the Borel set $\{(x,z) \in \mathcal{T} : d_{\mathcal{T}}(z,Y) < d_{\mathcal{T}}(x,Y)\}$ has countable sections, so we may find a Borel function $f : X \setminus Y \to X$ with $d_{\mathcal{T}}(f(x),Y) < d_{\mathcal{T}}(x,Y)$ for all $x \in X \setminus Y$. For each $x \in X$ let n be least with $f^n(x) \in Y$, and let $\pi(x) = f^n(x)$. The set $A = \{(x,z) \in \mathcal{T} : \pi(x) \neq \pi(z)\}$ is Borel, and the map $A \to (\mathcal{R}_G)_{\restriction Y}$, $(x,z) \mapsto (\pi(x),\pi(z))$ is injective since $\mathcal{T}$ is a tree. Thus, the graph $\mathcal{T}_Y = \{(\pi(x),\pi(z)) : (x,z) \in A\}$ is Borel, and $\mathcal{T}_Y$ is acyclic with $\mathcal{R}_{\mathcal{T}_Y} = (\mathcal{R}_G)_{\restriction Y}$ since $\mathcal{T}$ is acyclic with $\mathcal{R}_{\mathcal{T}} = \mathcal{R}_G$.

(3)⇒(2) is immediate from Theorem A.2. (2)⇒(1): Suppose that (2) holds and let $\mathcal{T}_Y$ be a Borel treeing of $(\mathcal{R}_G)_{\restriction Y}$. Since the set $\{(x,y) \in \mathcal{R}_G : x \in X \setminus Y,\ y \in Y\}$ has countable sections we may find a Borel map $f : X \setminus Y \to Y$. Then the graph $\mathcal{T} = \mathcal{T}_Y \cup \{(x,f(x)) : x \in X \setminus Y\} \cup \{(f(x),x) : x \in X \setminus Y\}$ is a Borel treeing of $\mathcal{R}_G$. □

Proposition A.7. *Let $G \curvearrowright X$ be a Borel action of a lcsc group G on a standard Borel space X and let μ be a G-quasi-invariant Borel probability measure on X. Let Y be a Borel cross section for the action. Then there exists a Borel probability measure ν on Y which is $(\mathcal{R}_G)_{\restriction Y}$-quasi-invariant, and satisfies $\nu(A) = 0$ if and only if $\mu([A]_{\mathcal{R}_G}) = 0$, for all Borel $A \subseteq Y$. Moreover, if ν' is any other $(\mathcal{R}_G)_{\restriction Y}$-quasi-invariant Borel probability measure on Y with this property then $\nu \sim \nu'$.*

Proof. Fix an open precompact neighborhood U of 1_G such that the map $U \times Y \to X$, $(g,y) \mapsto g \cdot y$, is injective. Since G is lcsc we may find a sequence $(g_n)_{n\in\mathbf{N}}$ in G such that $G = \bigcup_n g_n U$. Then $1 = \mu(X) = \mu(\bigcup_n g_n U \cdot Y)$, so $\mu(g_n U \cdot Y) > 0$ for some $n \in \mathbf{N}$ and hence $\mu(U \cdot Y) > 0$ since μ is quasi-invariant. Let μ_0 denote the normalized restriction of μ to $U \cdot Y$, and let ν be the Borel probability measure on Y given by $\nu(A) = \mu_0(U \cdot A)$. Suppose that $\nu(A) = 0$, so that $\mu(U \cdot A) = 0$. Since μ is G-quasi-invariant we have $\mu(g_n U \cdot A) = 0$ for all $n \in \mathbf{N}$ and hence $\mu([A]_{\mathcal{R}_G}) \leq \sum_n \mu(g_n U \cdot A) = 0$. Conversely, if $A \subseteq Y$ is a Borel set with $\mu([A]_G) = 0$, then $\nu(A) = \mu(U \cdot A)/\mu(U \cdot Y) \leq \mu([A]_{\mathcal{R}_G})/\mu(U \cdot Y) = 0$. It follows that ν is $(\mathcal{R}_G)_{\restriction Y}$-quasi-invariant. The last statement is clear. □

In the case of a free p.m.p. action of a unimodular lcsc G, the following is well known (see [KPV15] for a detailed treatment).

Proposition A.8. *Let $G \curvearrowright X$ be a free Borel action of a unimodular lcsc group G on a standard Borel space X and let μ be a G-invariant Borel probability measure on X. Fix a Haar measure λ on G. Let Y be a Borel cross section for the action. Then there is a unique $(\mathcal{R}_G)_{\restriction Y}$-invariant probability measure ν_Y on Y, and a unique value $0 < \mathrm{covol}(Y) < \infty$ such that, for any neighborhood U of 1_G as in the definition of cross section, the pushforward of $(\lambda_{\restriction U}) \times \nu_Y$ under the map $(g,y) \mapsto gy$ is equal to $\mathrm{covol}(Y)(\mu_{\restriction UY})$.*

Moreover, if Y' is any other cross section for the action, then there exist cross sections $Y_0 \subseteq Y$ and $Y_0' \subseteq Y'$, together with a measure preserving bijection $\varphi : (Y_0, \nu_{Y_0}) \to (Y_0', \nu_{Y_0'})$ taking $\mathcal{R}_{\restriction Y_0}$ to $\mathcal{R}_{\restriction Y_0'}$, such that

$$\frac{\nu_Y(Y_0)}{\nu_{Y'}(Y_0')} = \frac{\mathrm{covol}(Y)}{\mathrm{covol}(Y')}.$$

Recall, the **cost** of p.m.p. countable Borel equivalence relation R on (Y, ν) is defined to be the infimum of the ν_R-measures of generating sets for R, where ν_R is the natural Borel measure on R [Gab00]. The above proposition allows one to make sense of the following definition (see also [Car18, Definition 3.1]).

Definition A.9. Let G be a unimodular lcsc group, and fix a Haar measure λ on G. Let $G \curvearrowright X$ be a free Borel action of G on a standard Borel space X and let μ be a G-invariant Borel probability measure on X. The **cost** of $\mathcal{R}_G$ is defined by

$$\mathrm{cost}(\mathcal{R}_G) - 1 = \frac{\mathrm{cost}((\mathcal{R}_G)_{\restriction Y}) - 1}{\mathrm{covol}(Y)},$$

where Y is any cross section for the action.

The **cost** of (G, λ) is defined to be the infimum of costs of orbit equivalence relations generated by free p.m.p. actions of G. We say G has **fixed price** if all such orbit equivalence relations have the same cost (this is independent of the choice of Haar measure).

Example A.10. Let $G = \mathrm{Aut}(T_n)$ be the automorphism group of the n-regular tree T_n. Fix a vertex of T_n, let K denote its stabilizer in G, and let λ be the Haar measure on G giving K measure 1. For any free Borel action $G \curvearrowright X$, modding out by K delivers a treeable countable Borel equivalence relation $\mathcal{R}_0$. A Borel transversal Y for the action of K on X is then a cross section for the action of G, and $(\mathcal{R}_G)_{\restriction Y}$ is isomorphic to $\mathcal{R}_0$, and thus treeable. Moreover, if μ is any G-invariant Borel probability measure on X, then the relations $\mathcal{R}_0$ and $(\mathcal{R}_G)_{\restriction Y}$ are p.m.p. treeable of cost $n/2$, and $\mathrm{covol}(Y) = 1$. We conclude that $\mathrm{Aut}(T_n)$ is Borel strongly treeable, and that $(\mathrm{Aut}(T_n), \lambda)$ has fixed price $n/2$.

Proposition A.11. *Let G be a unimodular lcsc group that is strongly treeable. Then G has fixed price.*

Proof. Using cross sections (together with the fact that the pull back of a cross section of a free action under a factor map is a cross section) the proof follows as in [Gab00, Proposition VI.21]. □

The cost of some lcsc groups is also considered in [AM21].

B. Treeability of groups and permanence properties

Definition B.1. Let G be a locally compact second countable group.

(1) We say that G is **Borel strongly treeable** if every free Borel action of G is Borel treeable.

(2) We say that G is **measure strongly treeable** if every free Borel action of G is measure treeable.
(3) We say that G is **strongly treeable** if every free p.m.p. action $G \curvearrowright (X, \mu)$ of G is μ-treeable.
(4) We say that G is **treeable** if there exists some free p.m.p. action $G \curvearrowright (X, \mu)$ of G which is μ-treeable.

We renew our warning that "arborable" and "anti-arborable" in [Gab00] stands for what we call strongly treeable and non-treeable respectively here.

Let $\mathfrak{BST}$, $\mathfrak{MST}$, $\mathfrak{ST}$, and $\mathfrak{T}$ denote the classes of Borel strongly treeable, measure strongly treeable, strongly treeable, and treeable lcsc groups respectively. Then clearly

$$\mathfrak{BST} \subseteq \mathfrak{MST} \subseteq \mathfrak{ST} \subseteq \mathfrak{T}.$$

Question B.2. *Which of these containments, if any, are strict?*

Theorem B.3. *Let $\mathfrak{BST}$, $\mathfrak{MST}$, $\mathfrak{ST}$, and $\mathfrak{T}$ be as above. Then:*

(1) $\mathfrak{BST}$ contains all countable locally nilpotent groups and all locally compact compactly generated groups of polynomial growth.
(2) $\mathfrak{MST}$ contains all lcsc amenable groups.
(3) $\mathfrak{MST}$ contains $\mathrm{Isom}(\mathbf{H}^2)$, $\mathrm{SL}_2(\mathbf{R})$, and $\mathrm{PSL}_2(\mathbf{R})$.
(4) Let $\mathfrak{C}$ be one of $\mathfrak{BST}$, $\mathfrak{MST}$, or $\mathfrak{ST}$. Suppose that

$$1 \to K \to G \to L \to 1$$

is a short exact sequence where K is compact and $L \in \mathfrak{C}$. Then $G \in \mathfrak{C}$.
(5) $\mathfrak{BST}$, $\mathfrak{MST}$, and $\mathfrak{T}$ are closed under taking closed subgroups. If $G = G_1 * G_2$ is countable and belongs to $\mathfrak{ST}$ then so does G_1.
(6) $\mathfrak{ST}$ is closed under taking closed co-finite subgroups. In particular, if $G \in \mathfrak{ST}$ then every lattice of G is in $\mathfrak{ST}$.
(7) If G has a closed co-finite subgroup in $\mathfrak{T}$, then $G \in \mathfrak{T}$. In particular, if G has a lattice in $\mathfrak{T}$, then $G \in \mathfrak{T}$.
(8) $\mathfrak{BST}$, $\mathfrak{MST}$, $\mathfrak{ST}$ and $\mathfrak{T}$ are each closed under taking free products (of countable groups) with amalgamation over a finite subgroups.
(9) $\mathfrak{BST}$, $\mathfrak{MST}$, and $\mathfrak{ST}$ are each closed under taking HNN extensions (of countable groups) with respect to an isomorphism between two finite subgroups.
(10) Let $\mathfrak{C}$ be one of $\mathfrak{BST}$, $\mathfrak{MST}$, or $\mathfrak{ST}$. Suppose that G is the Bass-Serre fundamental group of a graph of countable groups in which each vertex group is in $\mathfrak{C}$, and each edge group is finite. Then $G \in \mathfrak{C}$.

(1) follows from [SS13] and [JKL02]. Part (2) follows from [CFW81]. Part (3) is Corollary 4.2. The proofs of the remaining facts are routine generalizations of results appearing in the literature: (4) follows immediately from the fact that Borel actions of compact groups are smooth, (5) generalizes [Gab00, Th. 5] and follows also from [JKL02, Proposition 3.3] (treeability for subrelations, together with a standard induction $H\backslash(X \times G) \curvearrowleft G$ – choosing of a finite equivalent measure when necessary – and restriction to the identity slice for $\mathfrak{BST}$ and $\mathfrak{MST}$); the "$\mathfrak{ST}$ part" requires the use of [Tör06] (any two free p.m.p. actions of G_1 and G_2 on (X, μ) can be realized

by the restrictions of a free p.m.p. action of $G_1 * G_2$ up to a conjugation of the G_2-action by an element of $\mathrm{Aut}(X, \mu)$). (6) generalizes [Gab00, Th. VI.19 (i)], (7) follows from a standard induction argument, and (8), (9) and (10) generalize [Gab00, Prop. VI.10] from the context of countable groups and p.m.p. actions (the proofs extend immediately to the Borel context; for $\mathfrak{T}$ in (8) use the diagonal action of the free product obtained from co-induction of treeable actions of the factors).

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