Errata: "Uniformity, Universality, and Computability Theory". The proof of Theorem 3.6 contains an error where a case in the argument was omitted. In this errata, we explain this error, and give a correct proof of Theorem 3.6.

In addition, there are two other minor errors in the paper which can be remedied as follows:

- In the paragraph preceding Theorem 4.5, the paper says that the isomorphism relation on the set contractible simplicial complexes of dimension $\leq n$ is Borel bireducible to the universal countable Borel equivalence relation structurable by contractible simplicial complexes of dimension $\leq n$. This is not the case (for examples, the isomorphism relation on locally finite trees is Borel bireducible to E_∞). The paper should instead say that the universal structurable Borel equivalence relation for the class of locally finite contractible simplicial complexes of dimension $\leq n$ is Borel bireducible with the isomorphism relation for *rigid* locally finite contractible simplicial complexes of dimension $\leq n$.
- In the proof of Theorem 3.3, the function g_i is never defined. It should be defined by $g_i(x) = \gamma_i \cdot x$, where γ_i is the ith generator of $\mathbb{F}_\omega$. In addition, the definition given of f is not correct. The definition should be that $f(x)(i) = c_i(x)$ if $c_i(x) \in \{0,1\}$. Otherwise if $c_i(x) = 2$, then
$$f(x)(i) = \begin{cases} 1 - c_i(g_i(x)) & \text{if } c_i(g_i^{-1}(x)) = c_i(g_i(x)) \\ c_i(g_i(x)) & \text{otherwise.} \end{cases}$$
To finish the proof of Theorem 3.3 after this definition, applying Theorem 3.2 yields an x so that $f(x)(i) = f(g_i(x))(i)$ for all i. From this, it follows that $c_i(x) = 2$ for all i.

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Fixing the error in Theorem 3.6

In the remainder of this note, we give a correct proof of Theorem 3.6. We begin the argument by noting that while the argument works fine with Lemma 3.5 as is, it is perhaps more logical to alter it as follows:

Lemma 3.5. Suppose that X is a standard Borel space, $g_0, g_1 \colon X \to X$ are partial Borel injections, and μ is a Borel probability measure on X. Then there is a μ-conull Borel set A and two Borel functions $c_0, c_1 \colon A \to 2$ such that for all $x \in X$, if $x \in \text{dom}(g_i)$ for all $i \in \{0,1\}$, and $g_i(x) \neq x$ for all $i \in \{0,1\}$, then there exists some $j \in \{0,1\}$ so that $c_j(x) \neq c_j(g_j(x))$.

Proof. By the proof of Lemma 3.5 in the paper. $\square$

Now we prove Theorem 3.6. Note that the set Z in Theorem 3.6 is used only for the proof of Theorem 3.9, and is not required to prove Corollary 3.1. If the reader is interested only in Corollary 3.1, they may safely set $Z = \emptyset$.

Theorem 3.6. Suppose Z is a countable set (which is possibly empty) which is disjoint from $\mathbb{F}_2 \times \omega$, and G is a countable group of permutations of the set $(\mathbb{F}_2 \times \omega) \sqcup Z$. Suppose also that for every $\delta \in \mathbb{F}_2$, there exists some $\rho_\delta \in G$ so that $\rho_\delta((\gamma, n)) = (\delta\gamma, n)$ for every $(\gamma, n) \in \mathbb{F}_2 \times \omega$. Then

(1) The permutation action of G on $2^{\mathbb{F}_2 \times \omega \sqcup Z}$ generates a measure universal countable Borel equivalence relation.
(2) The permutation action of G on $3^{\mathbb{F}_2 \times \omega \sqcup Z}$ generates a uniformly universal countable Borel equivalence relation.

Proof. Throughout we will let $Y \in \{2, 3\}$. Our construction for parts (1) and (2) will coincide until the very end when we split into cases.

Let E_∞ be a universal countable Borel equivalence relation generated by an action of $\mathbb{F}_2$ on a standard Borel space X. If $f \colon X \to Y^\omega$ is a function, then define $\hat{f} \colon X \to Y^{\mathbb{F}_2 \times \omega \sqcup Z}$ by

$$\hat{f}(x)((\gamma, n)) = f(\gamma^{-1} \cdot x)(n)$$

for $(\gamma, n) \in \mathbb{F}_2 \times \omega$, and $\hat{f}(x)(z) = 0$ for all $z \in Z$.

Note that if $x, y \in X$ and $\delta \cdot x = y$, then $\rho_\delta \cdot \hat{f}(x) = \hat{f}(y)$, since

$$\begin{aligned}\rho_\delta \cdot \hat{f}(x)((\gamma, n)) &= \hat{f}(x)(\rho_\delta^{-1}((\gamma, n))) = \hat{f}(x)((\delta^{-1}\gamma, n)) \\ &= f(\gamma^{-1} \cdot \delta \cdot x)(n) = \hat{f}(\delta \cdot x)((\gamma, n)) = \hat{f}(y)((\gamma, n))\end{aligned}$$

So given any Borel f, the associated $\hat{f}$ is a Borel homomorphism from E_∞ to the orbit equivalence relation of the permutation action of G on $Y^{\mathbb{F}_2 \times \omega \sqcup Z}$.

Precisely, to prove (2) we will define a Borel injection $f \colon X \to 3^{\mathbb{F}_2 \times \omega \sqcup Z}$ so that the corresponding $\hat{f}$ becomes our desired Borel reduction. To prove (1) we will show that for every Borel probability measure μ on X, there is a μ-conull Borel set $A \subseteq X$ and a Borel injection $f \colon A \to 2^{\mathbb{F}_2 \times \omega \sqcup Z}$ so that $\hat{f}$ is a Borel reduction of $E_\infty \restriction A$ to orbit equivalence relation of the permutation action of G on $2^{\mathbb{F}_2 \times \omega \sqcup Z}$.

Say that $\rho \in G$ is:

- Type I if there are infinitely many n such that $\rho^{-1}((1, n)) \in Z$.
- Type II if ρ is not type I and for every k, there exists $n, m > k$ with $n \neq m$ such that $\rho^{-1}((1, n)) \in \mathbb{F}_2 \times \{m\}$
- Type III if it is not type I or II, and there is some m so that there are infinitely many n such that $\rho^{-1}((1, n)) \in \mathbb{F}_2 \times \{m\}$.
- Type IV if it is not type I, II, or III. Hence, for all but finitely many n, $\rho^{-1}((1, n)) \in \mathbb{F}_2 \times \{n\}$.

For each $\rho \in G$ that is type III, fix some m_ρ such that there are infinitely many n such that $\rho^{-1}((1, n)) \in \mathbb{F}_2 \times \{m_\rho\}$. Then say that n **witnesses** ρ **is type III** if $\rho^{-1}((1, n)) \in \mathbb{F}_2 \times \{m_\rho\}$.

The rough idea of our proof is as follows. We will construct f in countably many steps. At each step, we will have some set $S \subseteq \omega$ and we will define $f(x)(n)$ for all $x \in X$ and $n \in S$. Our task is to ensure that for all ρ, either $\rho \cdot \hat{f}(x) \notin \operatorname{ran}(\hat{f})$, or $\rho \cdot \hat{f}(x) = \hat{f}(y)$ for some y such that $y \mathrel{E_\infty} x$. It will be very easy to diagonalize against ρ that are type I or II so that $\rho \cdot \hat{f}(x) \notin \operatorname{ran}(f)$. For ρ that are type III or IV, the argument is more complicated. For these ρ we first encode enough information into f so that for every $x \in X$, there is a unique $y = g_\rho(x)$ such that either $\rho \cdot \hat{f}(x) \notin \operatorname{ran}(\hat{f})$ or $\rho \cdot \hat{f}(x) = \hat{f}(y)$. If it is not the case that $y \mathrel{E_\infty} x$, we then diagonalize to ensure that $\rho \cdot \hat{f}(x) \neq \hat{f}(y)$. The most complicated case is when ρ is type IV. In this case the diagonalization requires some ideas from the theory of Borel graph colorings.

The error in the published version of the proof of this theorem essentially amounts to forgetting about type III permutations.

Let $\{S_0, S_1, S_2, S_3\}$ be a partition of ω so that S_2 and S_3 are infinite and for every $\rho \in G$,

- If ρ is type I, there is some $n \in S_1$ such that $\rho^{-1}((1,n)) \in Z$.
- If ρ is type II, there is some $n \in S_1$ such that $\rho^{-1}((1,n)) \in \mathbb{F}_2 \times \{m\}$ for some $m \in S_0$.
- if ρ is type III, then there are infinitely many $n \in S_2$ witnessing ρ is type III and infinitely many $n \in S_3$ witnessing that ρ is type III.

Such a partition can clearly be constructed inductively.

Our first part of the definition of f will be that for every $x \in X$,

$$f(x)(n) = 0 \land f(x)(m) = 1 \text{ for every } n \in S_0, \text{ and } m \in S_1. \tag{1}$$

Claim 1. If f is any function satisfying Equation 1 and ρ is type I or II, then for all $x \in X$, $\rho \cdot \hat{f}(x) \notin \operatorname{ran}(\hat{f})$

Proof of Claim. If ρ is type I, there is some $n \in S_1$ such that $\rho^{-1}((1,n)) = z$ for some $z \in Z$. Hence, $\rho \cdot \hat{f}(x)((1,n)) = \hat{f}(x)(\rho^{-1}((1,n))) = \hat{f}(x)(z) = 0$. But for all $y \in X$, $\hat{f}(y)((1,n)) = 1$ since $n \in S_1$. So $\rho \cdot \hat{f}(x) \notin \operatorname{ran}(f)$.

Similarly, if ρ is type II, there is some $n \in S_1$ such that $\rho^{-1}((1,n)) = (\gamma, m)$ for some $m \in S_0$ and $\gamma \in \mathbb{F}_2$. Hence, $\rho \cdot \hat{f}(x)((1,n)) = \hat{f}(x)(\rho^{-1}((1,n))) = \hat{f}(x)(\gamma, m) = 0$, since $m \in S_0$. But $\hat{f}(y)((1,n)) = 1$ for all $y \in X$ since $n \in S_1$. So $\rho \cdot \hat{f}(x) \notin \operatorname{ran}(f)$. $\square$

As a consequence of Claim 1, if $\rho \cdot \hat{f}(x) = \hat{f}(y)$ for some y, then it must be that both ρ and ρ^{-1} are type III or type IV.

Let $\rho_0, \rho_1, \ldots$ enumerate the group elements that are type III or IV and whose inverses are type III or IV. Let $S_{2,0}, S_{2,1}, \ldots$ be disjoint infinite subsets of S_2 so that

- if ρ_i is type III, then every $n \in S_{2,i}$ witnesses that ρ_i is type III
- if ρ_i is type IV, then for every $n \in S_{2,i}$, we have $\rho_i^{-1}((1,n)) \in \mathbb{F}_2 \times \{n\}$.

Say that a set $S \subseteq \omega$ is **good** if for every ρ that is type III, there are infinitely many $n \in S$ such that $\rho^{-1}((1,n)) \in \mathbb{F}_2 \times \{m_\rho\}$. So by the definition of the partition $\{S_0, S_1, S_2, S_3\}$, S_2 and S_3 must be good. We can construct such sets $S_{2,0}, S_{2,1}, \ldots$ since S_2 is good, and any good set can be partitioned into two good subsets.

For each i, let h_i be a Borel injection from $X \to 2^{S_{2,i}}$, and define

$$f(x)(n) = h_i(x)(n) \text{ for every } n \in S_{2,i} \tag{2}$$

Next, let $S'_{3,0}, S_{3,0}, S'_{3,1}, S_{3,1}, \ldots$ be disjoint subsets of S_3 so that

- If ρ_i or ρ_i^{-1} respectively are type III, then $S'_{3,i}$ contains m_{ρ_i} or $m_{\rho_i^{-1}}$ respectively if they are not contained in $S_0 \cup S_1 \cup S_2 \cup \bigcup_{j<i} S_{3,i} \cup S'_{3,i}$.
- If ρ_i is type III, then $S_{3,i}$ contains some n such that n witnesses ρ_i is type III.
- If ρ_i is type IV, then $S_{3,i}$ contains two elements, and for every $n \in S_{3,i}$ we have $\rho_i^{-1}((1,n)) \in \mathbb{F}_2 \times \{n\}$.

Such sets $S'_{3,0}, S_{3,0}, S'_{3,1}, S_{3,1}, \ldots$ can be constructed since S_3 is good, and any good set can be partitioned into two good subsets.

So that f is total, define

$$f(x)(n) = 0 \text{ for } n \in S_2 \setminus \bigcup_i S_{2,i} \text{ and } n \in S_3 \setminus \bigcup_i S_{3,i} \cup S'_{3,i}.$$

We will finish the remainder of the construction in countably many steps.

At step i, for all $x \in X$ we will define $f(x)(n)$ for $n \in S_{3,i}$ and $n \in S'_{3,i}$. To begin, for all $x \in X$, let

$$f(x)(n) = 0 \text{ for } n \in S'_{3,i}. \tag{3}$$

Claim 2. Suppose $\rho \in \{\rho_i, \rho_i^{-1}\}$. Then there is a fixed partial Borel function $g_\rho \colon X \to X$ so that for any Borel function f satisfying Equation 3 and the constraints on f imposed prior to step i of the construction, if $\rho \cdot \hat{f}(x) = \hat{f}(y)$, then $y = g_\rho(x)$.

Proof of Claim. Suppose first that ρ is type III. Let $\rho = \rho_j$. By Equation 3 and the definition of $S'_{3,i}$, $f(x)(m_\rho)$ has already been defined for every $x \in X$. Thus, for every $n \in S_{2,j}$, we have $\rho \cdot \hat{f}(x)((1,n)) = \hat{f}(x)(\rho^{-1}((1,n))) = \hat{f}(x)((\gamma, m_\rho)) = f(\gamma^{-1} \cdot x)(m_\rho)$ has already been defined. Thus, if $\rho \cdot \hat{f}(x) = \hat{f}(y)$ for some y, since $\hat{f}(y)((1,n)) = f(y)(n) = h_j(y)(n)$ for $n \in S_{2,j}$ and h_j is an injection, there is at most one y such that it could be the case that $\rho \cdot \hat{f}(x) = \hat{f}(y)$. Let $g_\rho \colon X \to X$ send x to this unique y if it exists. Precisely, $g_\rho(x) = h_j^{-1}(n \mapsto \rho \cdot \hat{f}(x)((1,n)))$.

Second, suppose ρ is type IV. Let $\rho = \rho_j$. By Equation 2, for $n \in S_{2,j}$, we have $(\rho \cdot \hat{f})(x)((1,n)) = f(\gamma^{-1} \cdot x)(n)$ for some $\gamma \in \mathbb{F}_2$. Thus, $(\rho \cdot \hat{f})(x)((1,n))$ has already been defined in the construction. Hence, there can be at most one y such that $(\rho \cdot \hat{f})(x)((1,n)) = \hat{f}(y)((1,n))$ for all $n \in S_{2,j}$, since h_j is an injection. Let $g_\rho(x)$ be this unique y if it exists. Precisely, let $g_\rho(x) = h_j^{-1}(n \mapsto (\rho \cdot \hat{f})(x)((1,n)))$. $\square$

Fix such functions g_{ρ_i} and $g_{\rho_i^{-1}}$. Let $g_i \colon X \to X$ be the partial Borel function where $g_i(x) = y$ if $g_{\rho_i}(x) = y$ and $g_{\rho_i^{-1}}(y) = x$. Then g_i is an injection. To finish the proof, it will suffice to finish the construction of f to show that either $g_i(x) \; E_\infty \; x$ or $\rho_i \cdot \hat{f}(x) \neq \hat{f}(g_i(x))$

Continuing our construction at step i, suppose first that ρ_i is type III. Fix $n_i \in S_{3,i}$, so by the definition of $S_{3,i}$, we have that n_i witnesses that ρ_i is type III. Let $\rho_i^{-1}((1,n_i)) = (\gamma_i, m_{\rho_i})$. Recall that $\rho_i \cdot \hat{f}(x)((1,n)) = f(\gamma_i^{-1} \cdot x)(m_{\rho_i})$ which has already been defined by Equation 3. We want to ensure that $\rho_i \cdot \hat{f}(x)((1,n_i)) \neq \hat{f}(g_i(x))((1,n_i)) = f(g_i(x))(n_i)$. So for all $x \in X$ define

$$f(x)(n_i) = 1 - f(\gamma_i^{-1} \cdot g_i^{-1}(x))(m_{\rho_i}) \text{ if } g_i^{-1}(x) \text{ is defined} \tag{4}$$

and $f(x)(n_i) = 0$ otherwise. Then the following is clear:

Claim 3. If ρ_i is type III, and f is any Borel function satisfying the constraints on f imposed so far in the construction, then for all x, $\rho_i \cdot \hat{f}(x) \notin \operatorname{ran}(\hat{f})$. $\square$

Continuing the construction at step i, suppose that ρ_i is type IV. For each $n \in S_{3,i}$, let $g_{i,n} \colon X \to X$ be the partial Borel injection $g_{i,n}(y) = \gamma_n^{-1} \cdot g_i^{-1}(y)$ where $\gamma_n \in \mathbb{F}_2$ is the group element such that $\rho^{-1}((1,n)) = (\gamma_n, n)$. Hence, if $g_i(x) = y$, then $\rho \cdot \hat{f}(x) = \hat{f}(y)$ would imply

$$\begin{aligned} f(y)(n) = \hat{f}(y)((1,n)) = (\rho \cdot \hat{f}(x))((1,n)) &= \hat{f}(x)(\rho_i^{-1}(1,n)) \\ = \hat{f}(x)((\gamma_n, n)) = f(\gamma_n^{-1} \cdot x)(n) &= f(g_{i,n}(y))(n) \end{aligned} \tag{5}$$

for every $n \in S_{3,i}$.

We now split into two cases and indicate how to finish the construction to prove parts (1) and (2) of Theorem 3.5.

We begin with part (1), continuing our construction at step i when ρ_i is type IV. Fix a Borel probability measure μ on X. Recall that $S_{3,i}$ has exactly two elements. By Lemma 3.5, there is a μ-conull Borel set $A_i \subseteq X$ and two Borel functions $c_{i,n} \colon A_i \to 2$ for $n \in S_{3,i}$ such that that for every $y \in A_i$, there is some $n \in S_{3,i}$ such that either $g_{i,n}(y) = y$, $g_{i,n}(y)$ is undefined, or $c_{i,n}(y) \neq c_{i,n}(g_{i,n}(y))$ We finish our definition of f by letting

$$f(y)(n) = c_{i,n}(y) \text{ for } n \in S_{3,i} \tag{6}$$

Then

Claim 4. For $x, y \in A_i$, if ρ_i is type IV, and $\rho_i \cdot \hat{f(x)} = \hat{f(y)}$, then $x \; E_\infty \; y$.

Proof of Claim. By our definition of g_i, we must have $y = g_i(x)$, and so $g_i^{-1}(y)$ is defined and hence $g_{i,n}(y)$ is also defined for all $n \in S_{3,i}$. If there is some n such that $g_{i,n}(y) = y$, then $x \; E_\infty \; y$ by the definition of $g_{i,n}$. Otherwise, $g_{i,n}(y) \neq y$ for $n \in S_{3,i}$. Hence, there is some $n \in S_{3,i}$ such that $c_{i,n}(y) \neq c_{i,n}(g_{i,n}(y))$ and hence $f(y)(n) \neq f(g_{i,n}(y))(n)$ which implies $\rho_i \cdot \hat{f}(x) \neq \hat{f}(y)$ by Equation 5. $\square$

To finish the proof of part (1) of Theorem 3.5, let $A = \bigcap A_i$. Since each A_i is conull, A is conull, and combining Claims 1, 3, and 4 proves that $\hat{f}$ is a Borel reduction of $E_\infty \upharpoonright A$ to the permutation action of G on $2^{\mathbb{F}_2 \times \omega \sqcup Z}$.

Now we show how to prove (2) in Theorem 3.5. For each $n \in S_{3,i}$, the function $g_{i,n}$ generates a Borel graph on X of degree at most 2. Let $c_{i,n} \colon X \to 3$ be a Borel 3-coloring of this graph by [Proposition 4.6, KST], and define

$$f(y)(n) = c_{i,n}(y) \text{ for all } n \in S_{3,i}. \tag{7}$$

The following claim has an identical proof to that of Claim 4.

Claim 5. For all $x, y \in X$, if ρ_i is type IV and $\rho_i \cdot f(\hat{x}) = f(\hat{y})$, then $x \mathrel{E_\infty} y$. $\square$

$\square$