A BAIRE CATEGORY PROOF OF THE ACKERMAN-FREER-PATEL THEOREM

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In this note, we give a proof of [AFP, Theorem 1.1] using the Baire category theorem. We also prove a slight generalization of [AFP, Theorem 3.19] where the underlying space is an arbitrary infinite Polish space instead of $\mathbb{R}$. Thanks to Colin Jahel for pointing out a serious error in a previous version of this note: in the proof of Lemma 0.1 were we defined the extension ordering of $\mathbb{P}$ incorrectly.

Suppose $\mathbb{A} = (A, R^A)_{R \in L}$ is a countable structure in a countable relational language L. Say $\mathbb{A}$ has **trivial definable closure** if for every finite tuple $\overline{a} \in A$, and for every $L_{\omega_1,\omega}$-formula $\phi(\overline{x}, y)$, if there is a unique $b \in \mathbb{A}$ such that $\mathbb{A} \vDash \phi(\overline{a}, b)$, then $b \in \overline{a}$. Equivalently, for all tuples $\overline{a}, \overline{b} \in A$, such that $\overline{a}$ and $\overline{b}$ are disjoint, there are infinitely many pairwise disjoint tuples $\overline{c} \in A$ such that $\mathrm{tp}^{\mathbb{A}}(\overline{a}, \overline{b}) = \mathrm{tp}^{\mathbb{A}}(\overline{a}, \overline{c})$ (see [Hod, 4.1.3]).

Lemma 0.1. *Suppose $\mathbb{A} = (A, R^{\mathbb{A}})_{R \in L}$ is a countable structure in a countable relational language L, where $\mathbb{A}$ has trivial definable closure. Then there exists a Borel L-structure $\mathbb{B} = (\omega^\omega, R^{\mathbb{B}})_{R \in L}$ on ω^ω (that is, the relations $(R^{\mathbb{B}})_{R \in L}$ are Borel) so that for any countable dense set $D \subseteq \omega^\omega$, $\mathbb{B} \restriction D$ is isomorphic to $\mathbb{A}$.*

Proof. By Morleyizing $\mathbb{A}$ (see [Hod, Section 2.6]) and expanding L, we may assume that there is a countable set T of Π_2 sentences in L such that if $\mathbb{C}$ is a countable structure, then $\mathbb{C} \vDash T$ if and only if $\mathbb{C}$ is isomorphic to $\mathbb{A}$. (After expanding the language this way and obtaining $\mathbb{B}$, take the reduct of $\mathbb{B}$ to the original language to obtain the desired structure).

If $s, t \in \omega^{<\omega}$ we write $s \subseteq t$ if s is an initial segment of t. We say s, t are incompatible if $s \nsubseteq t$ and $t \nsubseteq s$. We say that $S \subseteq \omega^{<\omega}$ is closed under initial segments if for all $t \in S$ and all $s \subseteq t$, $s \in S$. If $S \subseteq \omega^{<\omega}$ is finite and $t \in \omega^{<\omega}$, define $t \restriction S$ to be the maximal $s \in S$ so that $s \subseteq t$. So $t \restriction S$ is the longest initial segment of t that is in S. Similarly, if $x \in \omega^\omega$, define $x \restriction S$ to be the longest initial segment of x that is in S.

Let $\mathbb{P}$ be the set of finite partial injections from $\omega^{<\omega}$ to A whose domains are closed under initial segments. If $p, q \in \mathbb{P}$, say that q extends p if $q \supseteq p$ and for all pairwise incompatible strings $t_1, \ldots, t_n \in \mathrm{dom}(q)$, if $s_1 \restriction \mathrm{dom}(p), \ldots, s_n \restriction \mathrm{dom}(p)$ are pairwise incompatible, then

$$(*) \qquad \mathrm{tp}^{\mathbb{A}}(p(s_1 \restriction \mathrm{dom}(p)), \ldots, p(s_n \restriction \mathrm{dom}(p))) = \mathrm{tp}^{\mathbb{A}}(q(s_1), \ldots, q(s_n)).$$

That is, the type of $q(t_1), \ldots, q(t_n)$ has to be the same as the type of its "best approximation" in p, provided this best approximation is also a sequence of incompatible strings.

Let $Y \subseteq \mathbb{P}^\omega$ be the set of sequences $(p_i)_{i \in \omega}$ of elements of $\mathbb{P}$ so that if $i \leq j$, then p_j extends p_i, and so that $\bigcup(\mathrm{dom}(p_i)) = \omega^{<\omega}$. Note that Y is a G_δ subset of $\mathbb{P}^\omega$ and so is Polish.

We claim that since $\mathbb{A}$ has trivial definable closure, Y is nonempty. To see this, it suffices to show that if $p \in \mathbb{P}$ and $t \notin \mathrm{dom}(p)$ is such that the predecessor t^- of t is in $\mathrm{dom}(p)$, then we can extend p to $q \in \mathbb{P}$ where $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{t\}$. Let $r_1, \ldots, r_k$ be all the elements of $\mathrm{dom}(p)$ that are incompatible with t^- (note that these r_i are not necessarily pairwise incompatible). Since $\mathbb{A}$ has trivial definable closure, there is some $a \in \mathbb{A}$ that is not in $\mathrm{ran}(p)$ so that $\mathrm{tp}^{\mathbb{A}}(p(r_1), \ldots, p(r_k), a) = \mathrm{tp}^{\mathbb{A}}(p(r_1), \ldots, p(r_k), p(t^-))$. Let $q(t) = a$. We claim q extends p. Suppose $s_1, \ldots, s_n \in \mathrm{dom}(q)$ are pairwise incompatible. If $t \notin \{s_1 \ldots, s_n\}$, then (*) above is trivially satisfied since $s_i \restriction \mathrm{dom}(p) = s_i$ for every i. If $t \in \{s_1, \ldots, s_n\}$, then every s_i not equal to t cannot be compatible with t^- since $t \restriction \mathrm{dom}(p) = t^-$. Hence, (*) is satisfied by our choice of $q(t)$ since $\{s_1, \ldots, s_n\} \setminus \{t\}$ is a subset of the strings incompatible with t^-.

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Now each $(p_i)_{i\in\omega} \in Y$ yields a Borel L-structure $\mathbb{B}_{(p_i)} = (X, R^{(p_i)})_{R\in L}$ on X as follows. If $(x_1,\ldots,x_n)$ is an n-tuple in X, we define

$$R^{(p_i)}(x_1,\ldots,x_n) \leftrightarrow R^{\mathbb{A}}(p_i(x_1 \restriction \mathrm{dom}(p_i)),\ldots p(x_n \restriction \mathrm{dom}(p_i)))$$

for any sufficiently large i so that $x_j \neq x_k$ iff $x_j \restriction \mathrm{dom}(p_i)$ is incompatible with $x_k \restriction \mathrm{dom}(p_i)$. Roughly speaking, the type of $x_1,\ldots,x_n$ in $\mathbb{B}_{(p_i)}$ is determined by any p_i with a domain large enough to see which of the x_j are different. By the definition of extension in $\mathbb{P}$, note that truth value of $R^{\mathbb{B}_{(p_i)}}(p_i(x_1 \restriction \mathrm{dom}(p_i)),\ldots p(x_n \restriction \mathrm{dom}(p_i)))$ is the same for all such sufficiently large i. We claim that for every sentence φ in our Π_2 theory T, a comeager set of $(p_i) \in Y$ have the property that $(\mathbb{B}_{(p_i)} \restriction D) \vDash \varphi$ for any dense set $D \subseteq \omega^\omega$.

We may assume that every Π_2 sentence φ in our theory T has the form:

$$(\forall x_1,\ldots,x_n)[\bigwedge_{i\neq j} x_i \neq x_j \to (\exists y_1,\ldots,y_m)(\bigwedge_{i\neq j} y_i \neq y_j \wedge \theta(x_1,\ldots,x_n,y_1,\ldots,y_m))]$$

where θ is quantifier free. That is, φ says that for every pairwise distinct $x_1,\ldots,x_n$ there exists pairwise distinct $y_1,\ldots,y_m$ so that $\theta(x_1,\ldots,x_n,y_1,\ldots,y_m)$ is true.[1] Assuming that φ is in this form simplifies some of our book-keeping below. Fix such a Π_2 sentence φ and associated subformula θ.

The key claim is the following.

Claim. *Suppose $p \in \mathbb{P}$ is given and $r_1,\ldots,r_n$ are incompatible elements of $\mathrm{dom}(p)$. Then we claim there exists some q extending p so that if $s_1,\ldots,s_n \in \mathrm{dom}(q)$ are such that $s_i \supseteq r_i$ for all $i \leq n$, then there exists incompatible $t_1,\ldots,t_m \in \mathrm{dom}(q)$ so that $\mathbb{A} \vDash \theta(q(s_1),\ldots,q(s_n),q(t_1),\ldots,q(t_m))$.*

Proof of Claim. Let $(s_{i,1},\ldots,s_{i,n})_{i\leq k}$ be all n-tuples of extensions of $r_1,\ldots,r_n$ in $\mathrm{dom}(p)$. Let $(t_{i,j})_{i\leq k,j\leq m}$ be pairwise incompatible strings so that $t_{i,j} \restriction \mathrm{dom}(p)$ is the empty string for all i,j. For example, let all the $t_{i,j}$ be strings of length 1 whose first bit is sufficiently large.

Now define an injective q extending p where $\mathrm{dom}(q) = \mathrm{dom}(p) \cup \{t_{i,j} : i \leq k \wedge j \leq m\}$ by recursively finding $(t_{1,1},\ldots,t_{1,m}),\ldots,(t_{k,1},\ldots t_{k,m})$ so that $\mathbb{A} \vDash \theta(q(s_{i,1}),\ldots,q(s_{i,m}),q(t_{i,1}),\ldots,q(t_{i,m}))$. We can find such $q(t_{i,1}),\ldots,q(t_{i,m})$ so that q is an injection since $\mathbb{A}$ satisfies the formula φ and since $\mathbb{A}$ has trivial definable closure so there are infinitely many disjoint m-tuples witnessing the formula $\mathbb{A} \vDash \exists b_1,\ldots,b_m\theta(q(s_{i,1}),\ldots,q(s_{i,n}),b_1,\ldots,b_m)$.

Now since q is an injection, q is trivially an extension of p since all the elements $t \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$ have $t \restriction \mathrm{dom}(p)$ is the empty string. □ Claim.

Suppose $r_1,\ldots,r_n$, p, and q are as in the above claim. Then if $(p_i) \in Y$ is such that the sequence (p_i) contains q, then for any dense set $D \subseteq \omega^\omega$, for all $x_1,\ldots,x_n \in \omega^\omega$ extending $r_1,\ldots,r_n$, there exists $y_1,\ldots,y_m \in D$ so that $\mathbb{B}_{(p_i)} \vDash \theta(x_1,\ldots,x_n,y_1,\ldots,y_m)$. To see this, let $s_i = x_i \restriction \mathrm{dom}(q)$. Note that $s_i \supseteq r_i$. Then by the above claim, there are incompatible $t_1,\ldots,t_m$ so that $\theta(q(s_1),\ldots,q(s_n),q(t_1),\ldots,q(t_m))$. Now there must be proper extensions $t_1^*,\ldots,t_m^*$ of $t_1,\ldots,t_m$ so that $t_i^* \supsetneq t_i$, but $t_i^* \restriction \mathrm{dom}(q) = t_i$ (e.g. extend t_i to t_i^* so that its next bit is sufficiently large to not be in $\mathrm{dom}(q)$). Choose $y_1,\ldots,y_m \in D$ to be elements of $N_{t_1^*},\ldots,N_{t_m^*}$ (which must exist since D is dense). Then by the definition of $\mathbb{B}_{(p_i)}$, $\mathrm{tp}^{B_{(p_i)}}(x_1,\ldots,x_n,y_1,\ldots,y_m) = \mathrm{tp}^{A}(q(s_1),\ldots,q(s_n),q(t_1),\ldots,q(t_n))$, and hence $B_{(p_i)} \vDash \theta(x_1,\ldots,x_n,y_1,\ldots,y_m)$ by the above claim.

If $p \in \mathbb{P}$, define the open set $U_{p,n} = \{(p_i)_{i\in\omega} \in Y : p_n = p\}$. Note that these $U_{p,n}$ form a basis for Y. Now for each incompatible $r_1,\ldots,r_n$, the union of the set of $U_{q,k}$ where q satisfies the above claim is dense open by the above claim. Since any distinct $x_1,\ldots,x_n$ must extend some incompatible $r_1,\ldots,r_n \in \omega{<}\,\omega$, the set of $(p_i) \in Y$ so that $(\mathbb{B}_{(p_i)} \restriction D) \vDash \varphi$ for every dense set $D \subseteq \omega^\omega$ is comeager. Finally, since there are countably many $\varphi \in T$, this implies that the set of $(p_i) \in Y$ so that $(\mathbb{B}_{(p_i)} \restriction D) \vDash T$ for every dense set $D \subseteq \omega^\omega$ is comeager. □

[1] Given any Π_2 sentence $\forall x_1,\ldots,x_n\exists y_1,\ldots,y_m\psi(x_1,\ldots,x_n,y_1,\ldots,y_k)$, we can find an equivalent sentence in the desired form as follows. Let m be sufficiently large (e.g. $m = kn^n$) and have $\theta(x_1,\ldots,x_n,y_1,\ldots,y_m)$ be the formula $\theta(x_1,\ldots,x_n,y_1,\ldots,y_m) := \bigwedge_{\pi\colon \{1,\ldots,n\}\to\{1,\ldots,n\}} \bigvee_{\rho\colon \{1,\ldots,k\}\to\{1,\ldots,m\}} \theta(x_{\pi(1)},\ldots,x_{\pi(n)},y_{\rho(1)},\ldots,y_{\rho(k)})$. Then our original Π_2 sentence is equivalent to this Π_2 sentence in our desired form using the quantifier free formula θ in any structure that has infinitely many elements.

Recall that if L is a countable relational language, the space X_L is the set of all L-structures with universe ω. The group S_∞ of all permutations of ω acts on X_L by permuting the universe of each structure in X_L (see [K95, Section 16]).

Corollary 0.2 ([AFP, Theorem 1.1])**.** *Suppose $\mathbb{A} = (A, R^{\mathbb{A}})_{R \in L}$ is a countable structure in a countable relational language L. Then A has trivial definable closure if and only if there is an S_∞-invariant Borel probability measure μ on X_L that is supported on the set of structures isomorphic to $\mathbb{A}$.*

Proof. Suppose $\mathbb{A}$ has trivial definable closure. Let X be any perfect Polish space and let μ be an atomless Borel probability measure on X that assigns positive measure to every open subset of X. By Lemma 0.1, let $\mathbb{B} = (X, R^{\mathbb{B}})_{R \in L}$ be a Borel L-structure such that every countable dense set $D \subseteq X$ has $\mathbb{B} \upharpoonright D$ isomorphic to $\mathbb{A}$. Let μ^ω be the product probability measure on X^ω. Since μ is atomless and assigns positive measure to every open subset of X, μ^ω is supported on the set $Z \subseteq X^\omega$ of sequences $(x_i) \in X^\omega$ such that (x_i) is injective and dense in X. So each such (x_i) has $\mathbb{B} \upharpoonright \{x_i : i \in \omega\}$ isomorphic to $\mathbb{B}$.

Let $f\colon Z \to X_L$ be the function so that $f((x_i))$ is the structure on ω isomorphic to $\mathbb{B} \upharpoonright \{x_i : i \in \omega\}$ obtained by identifying x_i with i. Formally, $f((x_i)) = (\omega, R^{f((x_i))})_{R \in L}$ where

$$R^{f((x_i))}(n_0, \dots, n_k) \leftrightarrow R^{\mathbb{B}}(x_{n_0}, \dots, x_{n_k}).$$

Then the pushforward $f_* \mu^\omega$ of μ^ω under f is supported on the set of structures isomorphic to $\mathbb{A}$. This measure is S_∞-invariant because the permutation action of S_∞ on X^ω is μ^ω-invariant.

We now prove the converse. Suppose for a contradiction that $\mathbb{A}$ has nontrivial definable closure, but there exists an S_∞-invariant Borel probability measure μ on the set of structures in X_L isomorphic to $\mathbb{A}$. Let ϕ be an $L_{\omega_1,\omega}$ formula and $\overline{a} \in A$ be parameters so that $\mathbb{A} \vDash \exists! y \notin \overline{a} \phi(\overline{a}, y)$. If $\overline{n}$ is a tuple of elements of ω and $m \notin \overline{n}$, let $A_{\overline{n},m}$ be the set of structures $\mathbb{B} \in X_L$ isomorphic to $\mathbb{A}$ so that $\overline{n}$ is lexicographically least such that $\mathbb{B} \vDash \exists! y \notin \overline{n} \phi(\overline{n}, y)$, and m is the least element not in $\overline{n}$ such that $\mathbb{B} \vDash \phi(\overline{n}, m)$. The sets $A_{\overline{n},m}$ partition the set of models isomorphic to $\mathbb{A}$. So $\mu(\bigcup A_{\overline{n},m}) = 1$. However, if $m, m' \notin \overline{n}$, then $\mu(A_{\overline{n},m}) = \mu(A_{\overline{n},m'})$ since there is an element of S_∞ that fixes $\overline{n}$ but maps m to m'. We also have that $A_{\overline{n},m}$ and $A_{\overline{n},m'}$ are disjoint. Hence, since there are countably many $m \notin \overline{n}$ we must have $\mu(A_{\overline{n},m}) = 0$ for each $\overline{n}$, since μ is a probability measure. Thus, $\mu(\bigcup A_{\overline{n},m}) = 0$ which is a contradiction. □

We finish by noting that Lemma 0.1 can be generalized to find a Borel structure on an arbitrary infinite Polish space X so that its restriction to any countable dense subset is isomorphic to $\mathbb{A}$. First we need a trivial proposition about functions so that preimages of dense sets are dense.

Proposition 0.3. *If X is an infinite Polish space, then there is a Borel bijection f whose domain is a Borel subset of ω^ω and whose range is X so that if $D \subseteq X$ is dense, then $f^{-1}(D)$ is dense in ω^ω.*

Proof. Let $N_s = \{x \in \omega^\omega : x \subseteq s\}$ be the usual basis for ω^ω. Let $(s_n)_{n \in \omega}$ be an enumeration of $\omega^{<\omega}$. Let $(A_s)_{s \in \omega^{<\omega}}$ be disjoint uncountable Borel subsets of ω^ω so that $A_s \subseteq N_s$, and so that $\omega^\omega \setminus \bigcup_s A_s$ is uncountable. For example, define $A_{s_n} = \{x \in \omega^\omega : (\forall i \geq |s_n|) x(i) = 2n \vee x(i) = 2n+1\}$ where $|s_n|$ denotes the length of s_n. That is, A_{s_n} is the reals x so the every bit of x that occurs after the initial segment s_n is equal to $2n$ or $2n+1$.

Since X is infinite, there exists a countably infinite collection of disjoint open subset $(U_n)_{n \in \omega}$ in X. For each n, Let f_n be a bijection from a Borel subset of A_{s_n} to U_n. (Note that since U_n may be countable, the domain of f_n might need to be a proper subset of A_{s_n}). Now the domains $\operatorname{dom}(f_n)$ are disjoint since the A_s are disjoint. Let g be a Borel bijection between a Borel subset of $\omega^\omega \setminus \bigcup_n A_n$ and $X \setminus \bigcup_n U_n$.

Our desired function is $g \cup \bigcup_n f_n$. If $D \subseteq X$ is dense, then $f^{-1}(D)$ contains a point in N_{s_n} for every n. This is since there is some $x \in U_n$ so that $x \in D$ since D is dense and hence $f^{-1}(x) \in A_{s_n} \subseteq N_{s_n}$ by the definition of f and f_n. □

Corollary 0.4. *Suppose $\mathbb{A} = (A, R^{\mathbb{A}})_{R \in L}$ is a countable structure in a countable relational language L, where $\mathbb{A}$ has trivial definable closure. Then if X is an infinite Polish space, there exists a Borel L-structure $\mathbb{A}' = (X, R^{\mathbb{A}'})_{R \in L}$ on X (that is, the relations $(R^{\mathbb{A}'})_{R \in L}$ are Borel) so that for any countable dense set $D \subseteq X$, $\mathbb{A}' \restriction D$ is isomorphic to $\mathbb{A}$.*

Proof. Let f be a function as in Proposition 0.3, and let $\mathbb{B}$ be a Borel structure on ω^ω as in Lemma 0.1. Now let $\mathbb{A}'$ be the pushforward of $\mathbb{B}$ under f. That is, define $R^{A'}(x_1, \ldots, x_n) \leftrightarrow R^{\mathbb{B}}(f^{-1}(x_1), \ldots, f^{-1}(x_n))$. Since the inverse image of any dense set under f is dense in ω^ω, we are done. $\square$

References

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