IS THE TURING JUMP UNIQUE? MARTIN'S CONJECTURE, AND COUNTABLE BOREL EQUIVALENCE RELATIONS

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In 1936, Alan Turing wrote a remarkable paper giving a negative answer to Hilbert's Entscheidungsproblem [29]. Restated with modern terminology and in its relativized form, Turing showed that given any infinite binary sequence $x \in 2^\omega$, the set x' of Turing machines that halt relative to x is not computable from x. This function $x \mapsto x'$ is now known as the Turing jump, and it has played a singularly important role in the development of recursion theory, providing a canonical operator for increasing complexity in the Turing degrees.

In this essay, we shall present an overview of developments which have lead to a line of research with the potential for precisely explaining the central role that the Turing jump plays in recursion theory, and more generally in the theory of definability in mathematics. The centerpiece of this research direction is a conjecture of Martin which asserts in a very strong way the unique nature of the Turing jump. Although Martin's conjecture remains open, substantial partial results have been obtained which we view as strong evidence for its truth.

While Martin's conjecture paints a compelling picture, we will also discuss a conflicting possibility based on the difficulty of the problem of classifying the Turing degrees by invariants, as measured by the location of Turing equivalence in the hierarchy of countable Borel equivalence relations. A leitmotif of recursion theory is that degree structures such as the Turing degrees ought to be as rich and complex as possible. In the setting of Borel equivalence relations, the natural manifestation of this theme would be that Turing equivalence is a universal countable Borel equivalence relation. This would strongly contradict Martin's conjecture.

1. Basic definitions

We will work on the space 2^ω of infinite binary sequences. Such sequences can be canonically identified with subsets of ω. Given $x, y \in 2^\omega$, we say that x is Turing reducible to y, noted $x \leq_T y$, if there is a effective procedure for computing the bits of x when given y as an oracle. We say that x and y are Turing equivalent, noted $x \equiv_T y$, if $x \leq_T y$ and $y \leq_T x$. The equivalence classes of $\equiv_T$ are called the Turing degrees. Given $x, y \in 2^\omega$, we say that x is recursively enumerable relative to y if there is an effective procedure for enumerating the bits of x given y as an oracle. The eth recursively enumerable set relative to $x \in 2^\omega$ is noted W_e^x.

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2. The hierarchy of definability

Perhaps the first hint of the foundational and robust nature of the Turing jump was the connections that were discovered between the Turing jump and the hierarchy of definability in arithmetic. The arithmetical hierarchy was developed independently by Kleene [10] and Mostowski [18] during the Second World War. It ranks subsets of the natural numbers based on the complexity of the formulas in the language of arithmetic used to define them. The central relationship between the Turing jump and definability in arithmetic is the strong hierarchy theorem of Kleene and Post [12] which says that given any set $x \in 2^\omega$, a set $y \in 2^\omega$ has a Σ^0_{n+1} definition relative to x if and only if y is recursively enumerable in $x^{(n)}$, the nth iterate of the Turing jump of x. From this we can also deduce the corollary, originally due to Post, that y is Δ^0_{n+1} definable relative to x if and only if y is recursive in $x^{(n)}$. Note that the strong hierarchy theorem does not merely give a way of interpreting definability in arithmetic in terms of the Turing jump and computability. It also gives an alternate way of defining the Turing jump in terms of arithmetical definability – the Turing jump of x is the complete $\Sigma^0_1(x)$ set.

This connection between computability relative to the Turing jump and definability was quickly recognized as fundamental and was significantly refined and extended. Kleene [11] pushed this correspondence into the transfinite by devising a way to iterate the Turing jump through the countable ordinals. If $\alpha < \omega_1$ is an ordinal that has a representation recursive in x, then one can define $x^{(\alpha)}$, the αth iterate of the Turing jump of x, by taking the Turing jump at successor stages and taking an effective amalgamation at limit stages. Kleene then showed that the hierarchy theorem persists, relating computability in terms of transfinite iterates of the Turing jump with the hyperarithmetical hierarchy.

An even more striking connection was discovered between the Turing jump and definable subsets of 2^ω. Unlike the arithmetical hierarchy which was created in an effort to better understand the incompleteness phenomenon, the Borel hierarchy on subsets of 2^ω had been defined by the French intuitionists around the turn of the century, and seemed completely divorced from notions of computation. The Borel sets are defined by starting with the basic open subsets of 2^ω, and closing off under countable unions and complements in a transfinite induction of length ω_1. The Borel sets are widely regarded as containing all "concretely definable" subsets of 2^ω.

In the late 1950s, Addison [1], motivated by analogies between properties of the Borel sets and arithmetic sets, recognized that an effective version of the Borel hierarchy could be created. The central idea was that an effectively Borel set should be defined by a transfinite sequence of unions and complements that can be recursively presented. By relativizing to arbitrary oracles, one recovers the Borel hierarchy, but with a refined presentation based on the exact parameters needed to define such sets, and a measure of the recursion-theoretic complexity needed to describe them.

Addison further realized that this effective version of the Borel hierarchy was actually a variant of the hyperarithmetic hierarchy on subsets of 2^ω. Here, a set $A \subseteq 2^\omega$ has a Σ^0_α definition if there is a Σ^0_α sentence φ so that $x \in A$ if and only if $\varphi(x)$ is true. Since $x^{(\alpha)}$ is the complete $\Sigma^0_\alpha(x)$ set, this gives a way of defining the effective Borel hierarchy in terms of the Turing jump.

3. The recursively enumerable degrees, and a degree invariant solution to Post's problem

While these coincidences between definability and the Turing jump were viewed as a success for the emerging subject of recursion theory, the search for a deeper explanation for these connections was motivated by a different line of inquiry. The study of the Turing degrees of the recursively enumerable sets was ignited by Post's famous 1944 paper [21]. At the time, only two r.e. degrees were known: the degree of the recursive sets, and the degree of the halting problem. The central question was whether there were other r.e. degrees, and this came to be known as Post's problem. The existence of intermediate r.e. degrees was proved independently and nearly simultaneously by Friedberg [4] and Muchnik [19]. The key tool was to use what is now known as a priority argument to mediate the competing demands one encounters in the construction of such r.e. sets. Priority arguments proved to be the key to unlocking the structure of the r.e. degrees, and a rich theory soon developed based on these constructions.

A fundamental property of the Turing jump is its degree invariance. If x and y are Turing equivalent, then so are x' and y'. Essentially, if y can compute x, then questions about whether a computer program halts relative to x can be effectively transformed into questions about whether a related computer program halts relative to y. This nice property of the Turing jump is key to many of its useful applications in recursion theory. When relativized, the solutions of Friedberg and Muchnik to Post's problem do not have this degree invariance property because of a sensitivity of the flow of these constructions to how their requirements are met. In the second edition of his book, *Degrees of Unsolvability*, Sacks posed the seemingly innocuous question of whether there was a degree invariant solution to Post's problem, mirroring the degree invariance of the Turing jump. This question turned out to be remarkably deep, and remains open to this day.

Question 1 (Sacks [23]). *Is there an e so that for all $x, y \in 2^\omega$, we have that*

(1) $x <_T W_e^x <_T x'$, and
(2) if $x \equiv_T y$, then $W_e^x \equiv_T W_e^y$.

The first progress on this problem was made by Lachlan, who proved that any degree invariant solution to Post's problem can not be uniform in how its degree invariance is witnessed.

Theorem 2 (Lachlan [13]). *If e is a degree invariant solution to Post's problem, then there cannot exist a $u : \omega^2 \to \omega^2$ so that if $x \equiv_T y$ via (i, j) then $W_e^x \equiv_T W_e^y$ via $u(i, j)$.*

The Turing jump is uniform in Lachlan's sense. Thus, this theorem already demonstrates a unique feature of the Turing jump among all r.e. operators.

4. Infinite games and Martin measure

New light was shed on Sacks' question from an unexpected source. In the 1960s, set theorists began studying infinite two-player games with perfect information. The central question was whether one of the players in such a game must have a winning strategy, analogously to a famous classical result of Zermelo for finite games. Mycielski had shown that there must be games where neither player has a winning strategy assuming the ZF axioms. However, the individual moves in these games were rather complicated sets. Mycielski and Steinhaus [20] suggested studying a simple class of games where two players, I and II, alternate playing single bits $x_i \in \{0,1\}$ so that together they produce an infinite binary sequence $x = x_0 x_1 x_2 \ldots$:

$$\begin{array}{llllllll} \mathrm{I} & x_0 & & x_2 & & x_4 & & \ldots \\ \mathrm{II} & & x_1 & & x_3 & & x_5 & \ldots \end{array}$$

Before the game begins, a payoff set $A \subseteq 2^\omega$ is fixed. After the game is played, player I wins if x is in A, and player II wins otherwise. The game associated to A is said to be determined if one of the players has a winning strategy.

Even for this restricted class of games, it is straightforward to use the axiom of choice to construct a payoff set A so that there is not a winning strategy for either player in the associated game. Nevertheless, it remained a possibility that games with "simple" payoff sets might always be determined. For example, Gale and Stewart [6] had proved in 1953 that all games with open payoff sets were determined.

In the late 1960's, Martin discovered a remarkable consequence of determinacy. Say that a set $A \subseteq 2^\omega$ is Turing invariant if $x \in A$ and $y \equiv_T x$ implies $y \in A$. A *Turing cone* is a set of the form $\{x : x \geq_T y\}$ for some y.

Theorem 3 (Martin [15])**.** *If $A \subseteq 2^\omega$ is Turing invariant, and the game with payoff set A is determined, then either A contains a Turing cone, or the complement of A contains a Turing cone.*

As recounted in [17], Martin initially viewed this theorem as evidence for the existence of concretely definable undetermined sets, expecting that some simple property of Turing degrees could be found that would both hold and fail cofinally in the Turing degrees. However, a thorough investigation did not produce any properties of Turing degrees with this feature.

Several years later, Martin proved his celebrated theorem that all Borel sets are determined [16]. Significantly, Martin's cone theorem has turned out to be intimately connected to determinacy. For instance, by a result of Friedman [5], any proof of Borel determinacy must use the existence of ω_1 iterates of the powerset; a vast amount of set theoretic power compared to the tools generally used to prove theorems about Borel sets. However, a careful analysis of Friedman's proof shows that over a rather weak base theory, the determinacy of all Borel games is equivalent to the statement that every Turing invariant Borel set contains a Turing cone or is disjoint from a Turing cone.

Because the intersection of countably many Turing cones contains a Turing cone, the function μ on the σ-algebra of Borel Turing invariant sets defined by

$$\mu(A) = \begin{cases} 1 & \text{if } A \text{ contains a Turing cone.} \\ 0 & \text{otherwise.} \end{cases}$$

is a measure. It is known as Martin measure. While other concepts of size such as Lebesgue measure and Baire category have played an important role in recursion theory, Martin measure has turned out to be the appropriate notion for understanding the global nature of the Turing degrees. To a large extent, this reflects the fundamental role of relativization in the subject; the Turing degrees retain their essential character and structural properties when restricted to Turing cones.

5. Martin's conjecture

Spurred on by these new consequences of determinacy and inspired by Sacks' Question 1, Martin posed a conjecture giving an exact characterization of the definable functions on the Turing degrees. Say that a function $f : 2^\omega \to 2^\omega$ is *Turing invariant* if $x \equiv_T y$ implies $f(x) \equiv_T f(y)$. We state a weak version of Martin's conjecture that applies to Borel functions:

Conjecture 4 (Martin [9]). *Suppose $f : 2^\omega \to 2^\omega$ is a Turing invariant Borel function. Then either there exists a y so that $f(x) \equiv_T y$ on a Turing cone of x, or there is an $\alpha < \omega_1$ so that $f(x) \equiv_T x^{(\alpha)}$ on a Turing cone of x.*

Using determinacy as a tool, Slaman and Steel were able to generalize Lachlan's Theorem 2 to all Borel functions. Say that a Turing invariant function f is *uniformly Turing invariant* if there exists a $u : \omega^2 \to \omega^2$ so that if $x \equiv_T y$ via (i, j), then $f(x) \equiv_T f(y)$ via $u(i, j)$.

Theorem 5 (Slaman and Steel [27] [28]). *Martin's conjecture holds for all uniformly Turing invariant Borel functions.*

Martin's conjecture and Slaman and Steel's partial results inspired work on analogous conjectures for other notions of degree. In the early 1990s, Slaman and Steel considered a version where Turing equivalence is replaced by arithmetic equivalence, and the Turing jump is replaced with the function $x \mapsto x^{(\omega)}$. Slaman and Steel discovered the existence of many pathological uniformly arithmetically invariant functions, and hence this analogue of Martin's conjecture for arithmetic degrees is false [14]. Downey and Shore investigated the case of reducibilities stronger than Turing reducibility, and proved in unpublished work that the analogous version of Martin's conjecture is also false for most of these reducibilities. For instance, in the case of bounded Turing reducibility, there is a uniformly bT-invariant solution to Post's problem on a bT-cone. One can view these failed analogies as confirming the exceptional and fundamental nature of Turing equivalence and of the Turing jump.

Slaman and Steel's result was also used to shed more light on Sacks' original Question 1. Using Theorem 5, Downey and Shore [3] showed that any degree invariant solution to Post's problem must be low_2 or high_2.

The case of non-uniform Turing invariant functions appears to be considerably more difficult than the uniform case. Slaman and Steel's proof of Theorem 5 is a pure determinacy argument and this abstract approach does not seem to work to analyze non-uniform functions. Slaman and Steel were able to prove two more cases of Martin's conjecture, but using more concrete tools from recursion theory. The first case was for recursive functions and their approach was to use an argument analyzing rates of convergence. The second case was for all Borel increasing, order-preserving functions. That is, functions f so that $f(x) \geq_T x$ for all x, and $x \geq_T y \to f(x) \geq_T f(y)$ for all x and y. Here they used a generalized version of the Posner-Robinson theorem for iterates of the Turing jump through ω_1 to prove that all such functions obey Martin's conjecture.

More recently, Slaman has formulated and proved a version of Martin's conjecture for all Borel functions from 2^ω to $\mathcal{P}(2^\omega)$, the powerset of 2^ω. A *closure operator* is a function $M : 2^\omega \to \mathcal{P}(2^\omega)$ satisfying:

(1) $x \in M(x)$ for all $x \in 2^\omega$.
(2) $M(x)$ is closed under join and Turing reducibility.
(3) If $x \leq_T y$, then $M(x) \subseteq M(y)$.

Slaman was able to exactly characterize all Borel closure operators up to a Turing cone in terms of the Turing jump.

Theorem 6 (Slaman [26]). *Every Borel closure operator is equal to one of the following three types of operators on some Turing cone:*

(1) $x \mapsto \{y : y \leq_T x^{(\alpha)}\}$ *where* $\alpha < \omega_1$.
(2) $x \mapsto \{y : y \leq_T x^{(\alpha)}$ *for some* $\alpha < \lambda\}$ *where* $\lambda < \omega_1$.
(3) $x \mapsto 2^\omega$.

Slaman's proof of Theorem 6 is reminiscent of Slaman and Steel's analysis of increasing, order-preserving functions. Notably, it relies on a more recent sharpening of the generalized Posner-Robinson theorem that is due to Shore and Slaman [25]. Of all the results surrounding Martin's conjecture, Slaman's Theorem 6 is perhaps the most impressive, achieving in this setting Martin's goal of confirming the unique and fundamental nature of the hierarchy of definability studied by recursion theorists.

An interesting source of closure relations comes from any natural notion of hypercomputation more powerful than Turing computation. Here, by natural, we mean that such a notion should be able to be relativized to an oracle, and that it ought to be closed under amalgamations and compositions with recursive functions. If for each $x \in 2^\omega$, we let $H(x) \subseteq 2^\omega$ be the sets that can be hypercomputed from the oracle x, then H satisfies the definition of a closure operator based on our naturalness assumptions. Hence, Slaman's theorem establishes that up to Martin measure, any nontrivial and natural Borel model of hypercomputation is really Turing computation relative to an iterate of the Turing jump, or a union of such models.

6. Countable Borel equivalence relations

A *Borel equivalence* relation on 2^ω is an equivalence relation on 2^ω that is Borel as a subset of $2^\omega \times 2^\omega$, or equivalently, has a $\mathbf{\Sigma}^0_\alpha$ definition for some $\alpha < \omega_1$. A Borel equivalence relation is said to be *countable* if all its equivalence classes are countable. Most equivalence relations from recursion theory, such as Turing equivalence, are countable Borel equivalence relations.

The standard way of comparing complexity among Borel equivalence relations is Borel reducibility. If E and F are Borel equivalence relations on 2^ω, then E is said to be *Borel reducible* to F, noted $E \leq_B F$, if there is a Borel function $f : 2^\omega \to 2^\omega$ so that $xEy \leftrightarrow f(x)Ff(x)$. Such a function induces an injection $\hat{f} : 2^\omega/E \to 2^\omega/F$. Borel reducibility is often motivated in terms of classification by invariants, where if $E \leq_B F$, then we can reduce the classification problem on E to the classification problem on F. Dually, we can also view the equivalence classes of a countable Borel equivalence relation as specifying a type of information, and Borel reducibility as comparing the complexity of equivalence relations in terms of the types of information they can encode. For example, the type of information represented by Turing degrees is associated to the equivalence relation of Turing equivalence.

The hierarchy of countable Borel equivalence relations is highly nontrivial. In particular, even though every countable Borel equivalence relation on 2^ω has exactly continuum many classes, this fact cannot be witnessed by Borel functions in general. For instance, classical ergodicity arguments can be used to show that Turing equivalence is not Borel reducible to the relation of equality on 2^ω.

An important structural property of the hierarchy of countable Borel equivalence relations is the existence of universal countable Borel equivalence relations, as established by Dougherty, Jackson and Kechris [2]. A countable Borel equivalence relation E is said to be *universal* if for all countable Borel equivalence relations F, we have $F \leq_B E$.

A major theme of recursion theory is that its degree structures are often as rich and complicated as possible. Many natural conjectures in the subject run along these lines. This viewpoint evolved, particularly in the 1970s and 1980s, from earlier, more naïve intuitions. For example, at the beginning of the investigation of the r.e. degrees, Schoenfield conjectured [24] that the r.e. degrees were an ω-saturated uppersemilattice. This would imply that the theory of the r.e. degrees admits quantifier elimination, and would hence be decidable. Schoenfield's conjecture was quickly refuted based on the existence of a minimal pair of r.e. degrees, but it took several years before it was realized how completely wrong the conjecture was. The theory of the r.e. degrees is in fact as undecidable as possible, being recursively isomorphic to the theory of true arithmetic.

Adopting this perspective of expecting significant complexity in degree structures, it would be very natural to conjecture a positive answer to the following question of Kechris:

Question 7 (Kechris [8])**.** *Is Turing equivalence a universal countable Borel equivalence relation?*

In terms of the intuitions outlined above, a positive answer would mean that the notion of information given by the Turing degrees is universal in the sense of being able to effectively encode any other kind of information, and as difficult as possible to classify by invariants. Supporting this perspective is the fact that several other equivalence relations from recursion theory are universal countable Borel equivalence relations. For example, polynomial time equivalence is universal, and Slaman and Steel [14] have shown that arithmetic equivalence is also universal. However, if Turing equivalence is universal, this would contradict Martin's conjecture; if Martin's conjecture is true, then it is easy to see that there is no Borel reduction from two disjoint copies of Turing equivalence to a single copy of Turing equivalence.

There is an intriguing open question of Hjorth that is related to the problem of what countable Borel equivalence relations from recursion theory are universal.

Question 8 (Hjorth [7])**.** *Suppose E and F are countable Borel equivalence relations, E is universal, and $E \subseteq F$. Must F be universal?*

A positive answer to Hjorth's question would imply that essentially all equivalence relations from recursion theory are universal; there are very simple universal countable Borel equivalence relations generated by recursive permutations, and most equivalence relations from recursion theory are supersets of recursive isomorphism. Indeed, early in the history of recursion theory, and inspired by the Erlangen program, Rogers [22] made the influential suggestion that recursion theory ought to study only those structures and properties that are invariant under recursive isomorphism.

There is a plausible but false intuition for why Hjorth's conjecture might be true. To show that a countable Borel equivalence relation E is universal, we must produce a Borel reduction of a universal countable Borel equivalence relation E_∞ to E. Such a Borel reduction f is generally constructed in two steps. First, a method of coding is chosen to ensure that if $x E_\infty y$ then $f(x) E f(y)$. Second, subject to these coding requirements, f is constructed so that $\neg x E_\infty y$ implies $\neg f(x) E f(y)$. A plausible attempt to prove Hjorth's conjecture would be to show that if $E \subseteq F$, and the universality of E can be witnessed by a particular coding method, then the universality of F can also be witnessed by the same coding method, by simply taking a more "generic" reduction f. This strong version of Hjorth's conjecture has been refuted in unpublished work of Montalbán, Reimann and Slaman. Their proof uses Slaman and Steel's partial results on Martin's conjecture in Theorem 5 to show that the universality of Turing equivalence cannot be witnessed in a uniform manner. However, there are universal subequivalence relations of Turing equivalence whose universality can be witnessed uniformly. Their results suggest that a positive answer to Kechris' question, or especially Hjorth's question would require considerable subtlety.

We end, then, with this possibility of a positive answer to Kechris' question as an alternative to Martin's conjecture. These two options inhabit opposite ends of a spectrum of possible conceptions of the Turing degrees, as quantified by the number of Turing invariant functions modulo Martin measure. While Martin's conjecture gives a precise and simple classification of all such functions, if the Turing degrees are a universal countable Borel equivalence relation, then there is a vastly larger universe of degree invariant constructions as yet uninvestigated in recursion theory.

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