Definability and Randomness - a Travelogue

Tedfest 2024

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Algorithmic randomness

Glossary

- Martin-Löf tests are a versatile framework to define individual random elements in a set.

- Premeasure $\rho : 2^{<\omega} \to [0,\infty)$
  Examples : $\lambda(\sigma) = 2^{-|\sigma|}$, $\mathscr{H}^s(\sigma) = 2^{-s|\sigma|}$

- ML test: c.e. set W such that for all n,
$$\sum_{\langle\sigma,n\rangle\in W} \rho(\sigma) \leq 2^{-n}$$

- $X \in 2^\omega$ is ρ-random if there does not exist a test W such that
$$\{X\} \subseteq \bigcap_n \bigcup_{\langle\sigma,n\rangle\in W} N_\sigma$$

Algorithmic randomness

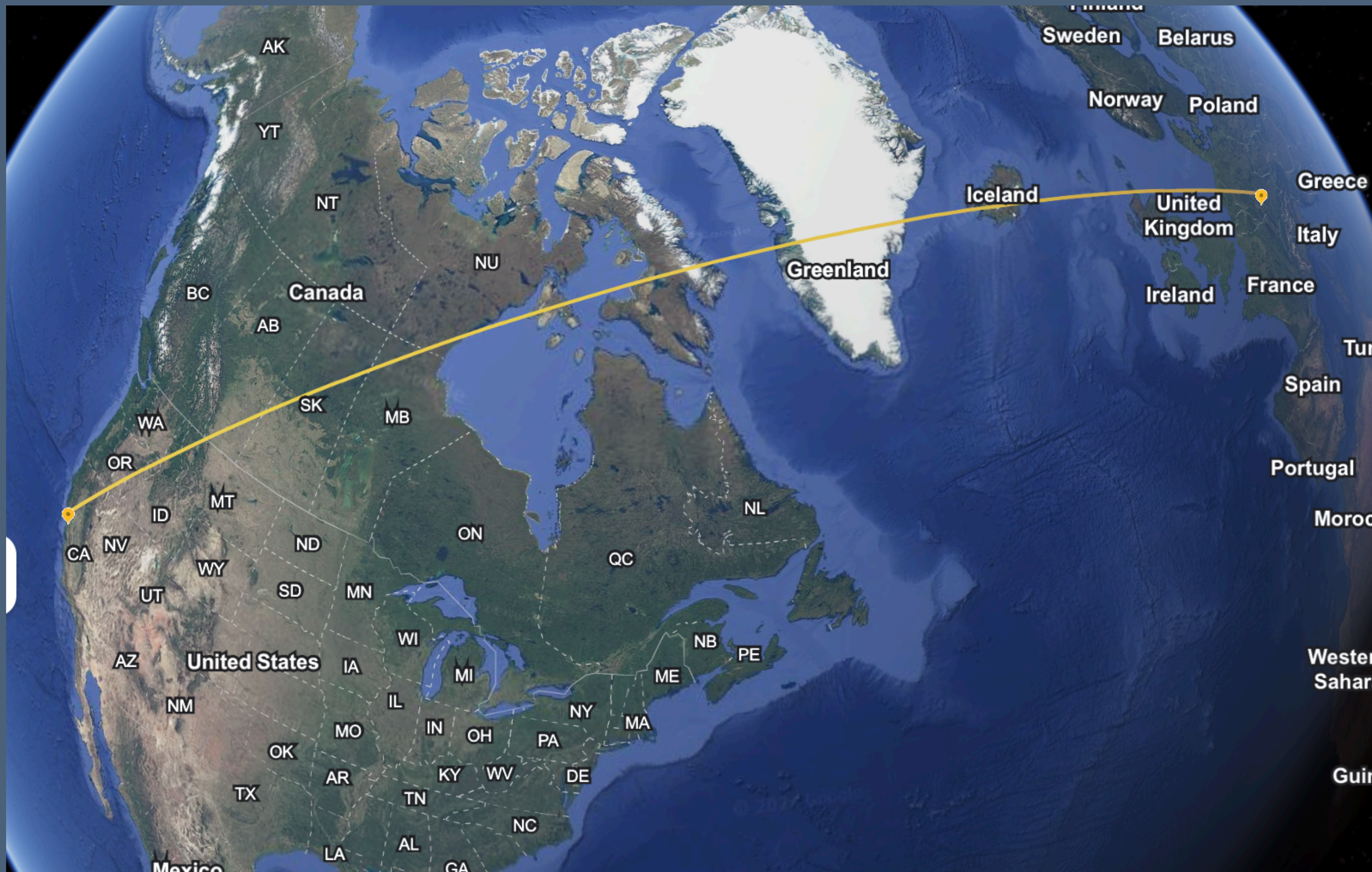
Glossary

- ML-random = λ-random
- $\dim_H(X) = \inf\{s \in \mathbb{Q}^+ : X$ not $\mathscr{H}^s$-random$\}$
- The framework can be adapted in order to work relative to oracles.
- n-randomness: random relative to $\varnothing^{(n-1)}$.

Heidelberg

2002-2003

- Extracting randomness: Can every real of effective dimension > 0 compute a ML-random real?
- Levin [1970]: If X is random for a computable measure and not computable, it is Turing equivalent to a ML-random sequence.
- Which sequences are random with respect to some (computable) measure?



Heidelberg -> Berkeley, 5780 miles

Berkeley

2004

- Turn this around: If X is Turing equivalent to a ML-random, then one can use it to *push forward* Lebesgue measure and make X random without making X an atom of the push-forward measure.
- Kucera-Gacs: Every real $\geq_T \varnothing'$ is Turing equivalent to a ML-random.
- Posner-Robinson: If A is not recursive, then there exists a G such that $A \oplus G \geq_T G'$.
- The push-forward needed an interesting lowness result for Π^0_1 -classes (of measures).

Berkeley -> Cordoba, 6064 miles

CCR 2004
1st International Conference on Computability, Complexity and Randomness, Córdoba, Argentina, 2004

Berkeley - Argentina

2004

- THM: For any real $X \in 2^\omega$, the following are equivalent:

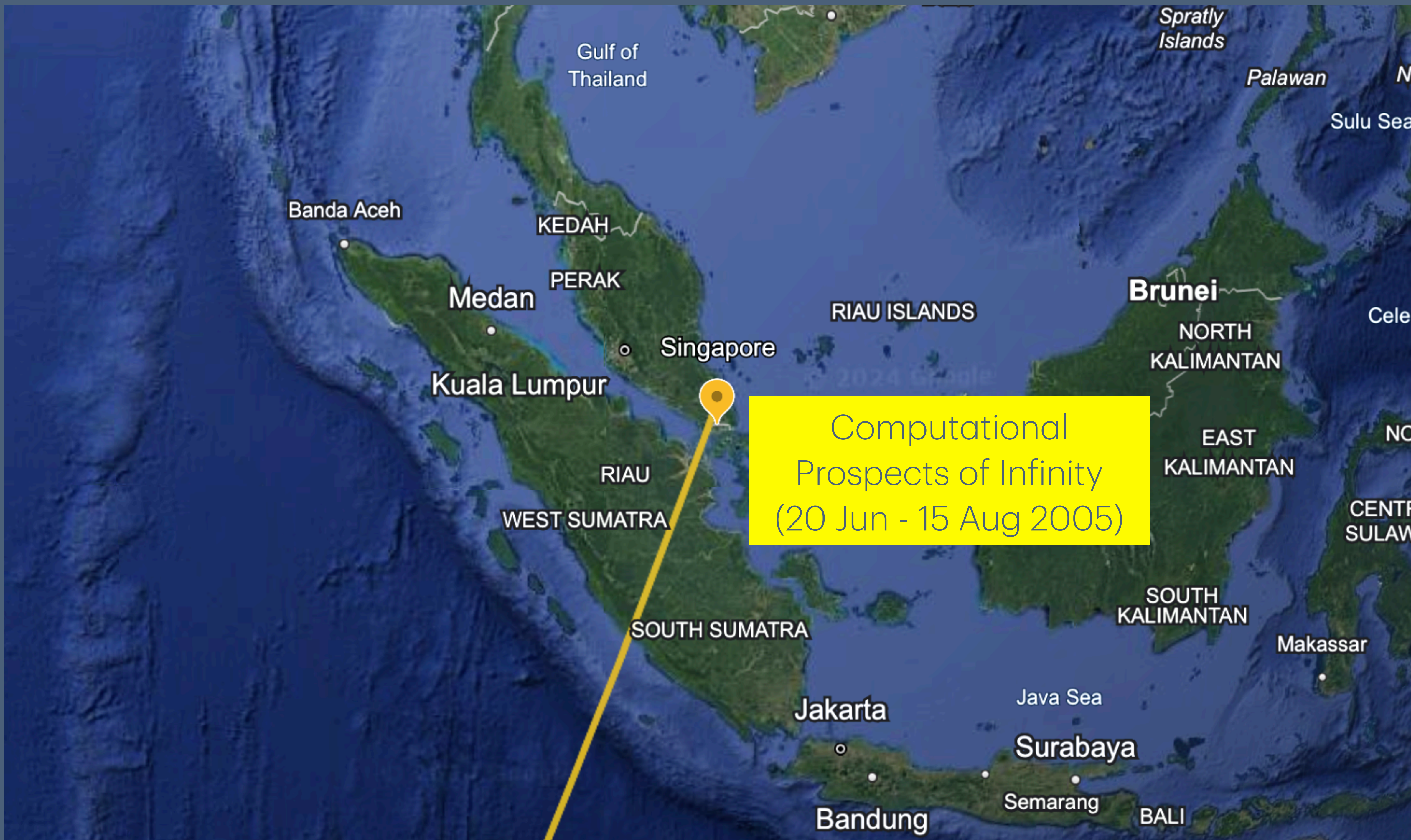
  (i) There exists a probability measure μ such that X is not a μ-atom and X is μ-random

  (ii) X is not computable.

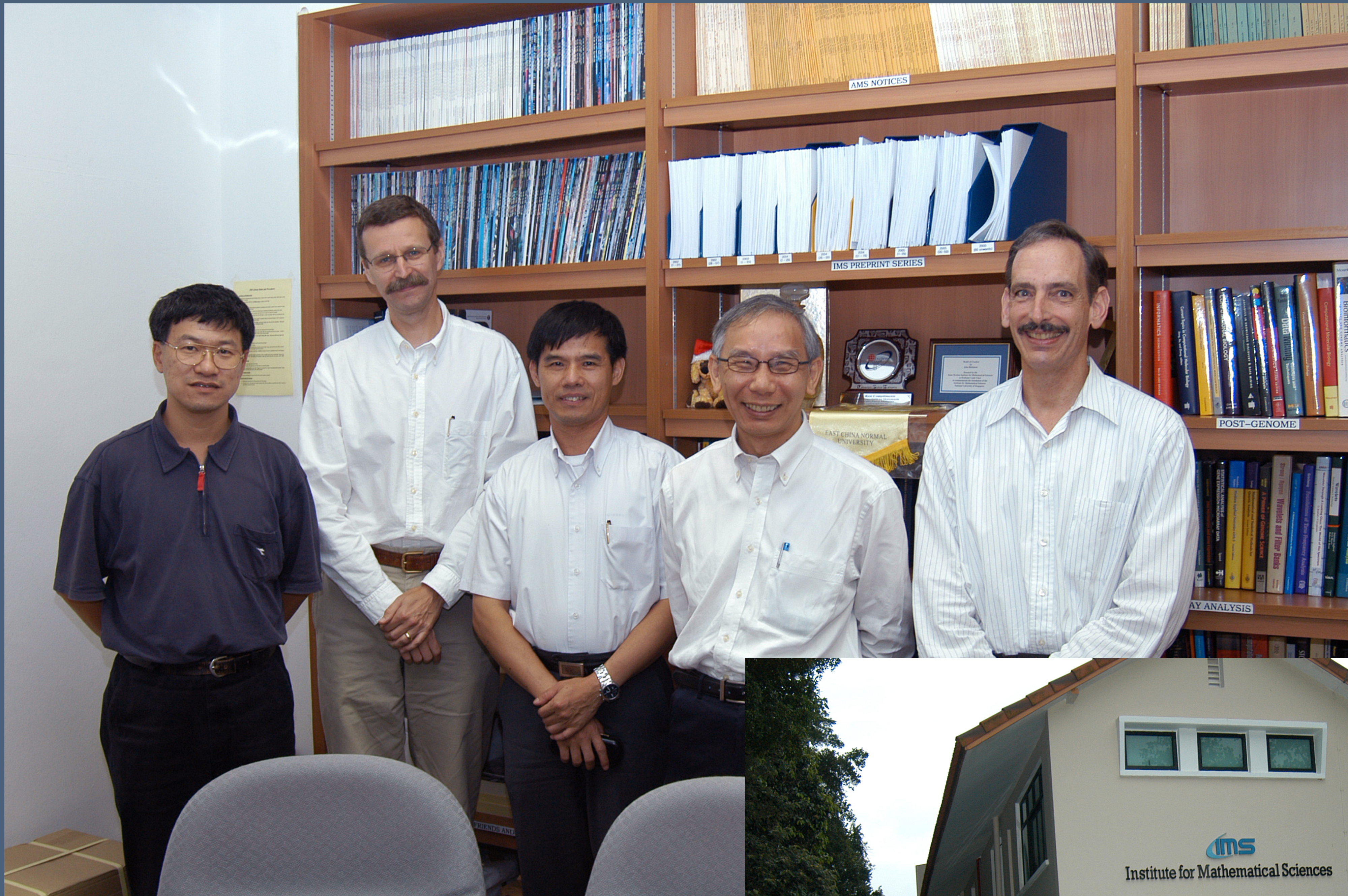
Argentina

2004

- In general, μ will still have atoms away from X.
- Can this be avoided?
- Using the settling function of the halting problem, one can construct a real $\equiv_T \varnothing'$ that is not random with respect to any non-atomic measure.
- This lead to the definition of
  $\mathrm{NCR} = \{X \in 2^\omega : X$ not random for any continuous measure$\}$



Cordoba - Singapore, 10322 miles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martin

In the meantime...

- $\mathbf{NCR}$ is a Π^1_1 set without a perfect subset.
- This puts it in L.
- Is it countable?
- Kjos-Hanssen and Montalban: Every member of a countable Π^0_1 class is in $\mathbf{NCR}$.
- This yields examples all the way through $\mathbf{HYP}$.

Singapore

2005

- Characterization of randomness with respect to *some* continuous measure: *being truth-table equivalent to a ML-random real.*

- Woodin: If X is not hyperarithmetic, then there is a Z such that $X \equiv_{tt(Z)} Z'$.

- Hence $\mathbf{NCR} \subseteq \mathbf{HYP}$ and $\mathbf{NCR}$ is co-final in the hyperarithmetic Turing degrees.

Singapore-Berkeley-Heidelberg

2005-2006

- Climbing the randomness ladder: What about n-randomness?
- Continuous randomness can be "pushed" via Turing reductions but reduces the level of randomness.
- We can use Borel determinacy to get a cone of continuous randomness:

  $\{X\colon \exists Z, R\ X \equiv_T Z \oplus R,\ R$ is $(n+3)$-random relative to $Z\}$

  is Turing invariant and co-final in the Turing degrees.
- It follows that the complement of $\mathbf{NCR}_n$ contains a Turing cone.

Singapore-Berkeley-Heidelberg

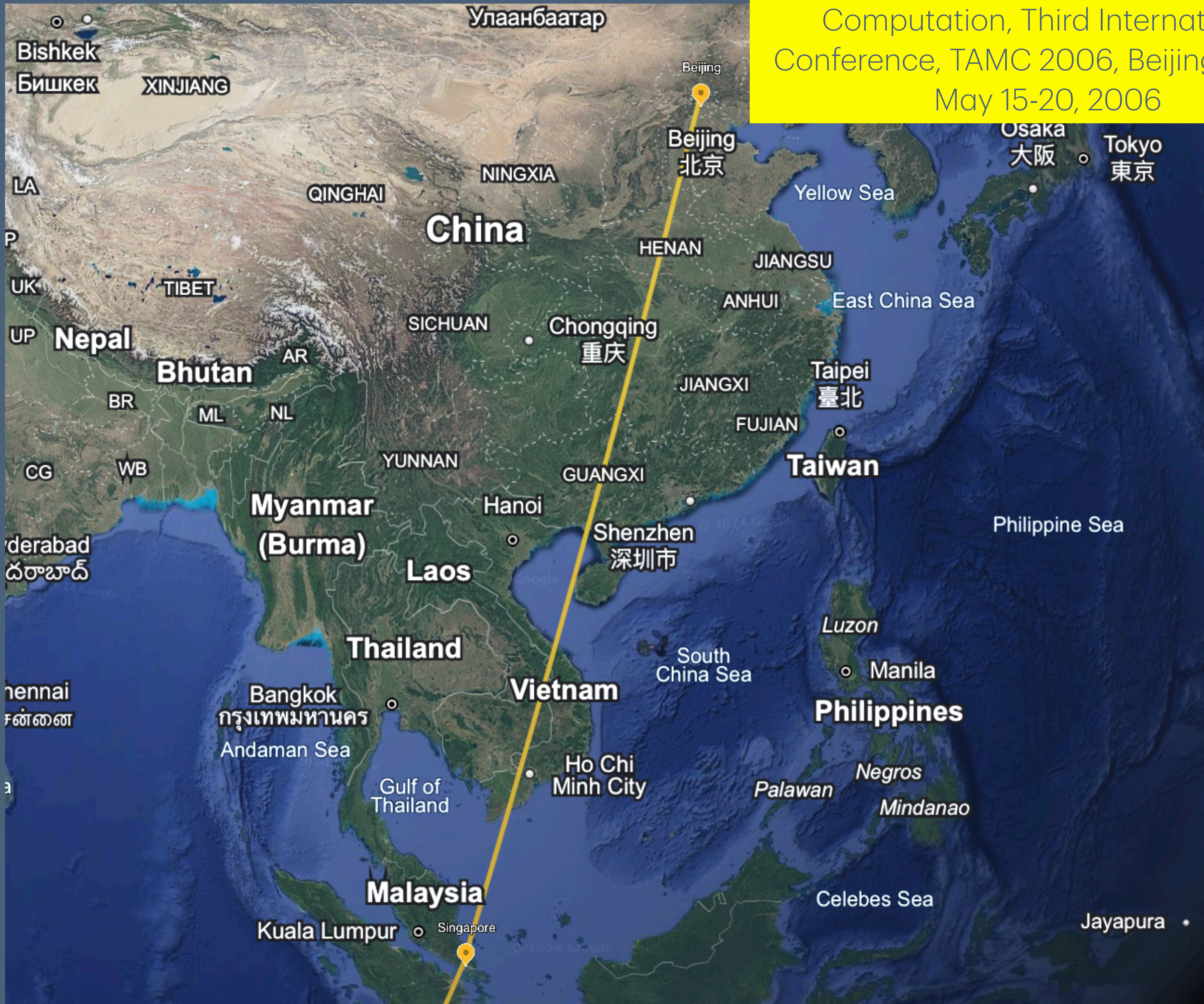
- A "higher" analogue of Posner-Robinson: Kumabe-Slaman forcing.
- β_n least ordinal such that $L_{\beta_n} \models \mathsf{ZFC}_n^-$
- LEMMA: Suppose that $n > 0$ and $X \in 2^\omega$ is not in L_{β_n}. Then there exists a real Φ such that $L_{\beta_n}[\Phi]$ is a model of ZFC_n^- and every real in $L_{\beta_n}[\Phi]$ is Turing reducible to $X \oplus \Phi$.
- In particular, X is in the cone above the winning strategy for the randomness pushing game (relative to Φ), and hence random for a continuous measure.

Singapore-Berkeley-Heidelberg

- THM: For all $n > 0$, $\mathbf{NCR}_n$ is countable.

- Friedman used the models L_{β_n} to show that the existence of iterates of the power set of ω is necessary to prove Borel determinacy.
- Could we do something similar to show iterated power sets of ω are necessary to prove the countability of NCR_n?

Theory and Applications of Models of Computation, Third International Conference, TAMC 2006, Beijing, China, May 15-20, 2006



Singapore-Beijing, 2772 miles

Beijing

2006

- *StairMaster technique:*

  Suppose μ is a continuous measure and Y is μ-n-random, $n \geq 2$. If $X \leq_T \mu^{(n-1)}$ and $X \leq_T Y \oplus \mu$, then $X \leq_T \mu$.

- One can use this to show that for any $k \geq 0$, if $X \equiv_T \varnothing^{(k)}$, then X is not 2-random with respect to any continuous measure.

- Using Enderton & Putnam's bound on uniform limits, one can extend this to $\varnothing^{(\omega)}$:

  If$X \equiv_T \varnothing^{(\omega)}$, then X is not 3-random with respect to a continuous measure.

Beijing

2006

- How can we extend this all the way through L_{β_n}?
- Use Jensen's master codes: canonical countings M_α of L_α with each M_α definable in a simple way from the sequence of its predecessors (like the Turing jump).
- The set $\{M_\alpha : \alpha < \beta_n\}$ is not countable in L_{β_n}.
- We want to apply a Stairmaster argument and show that

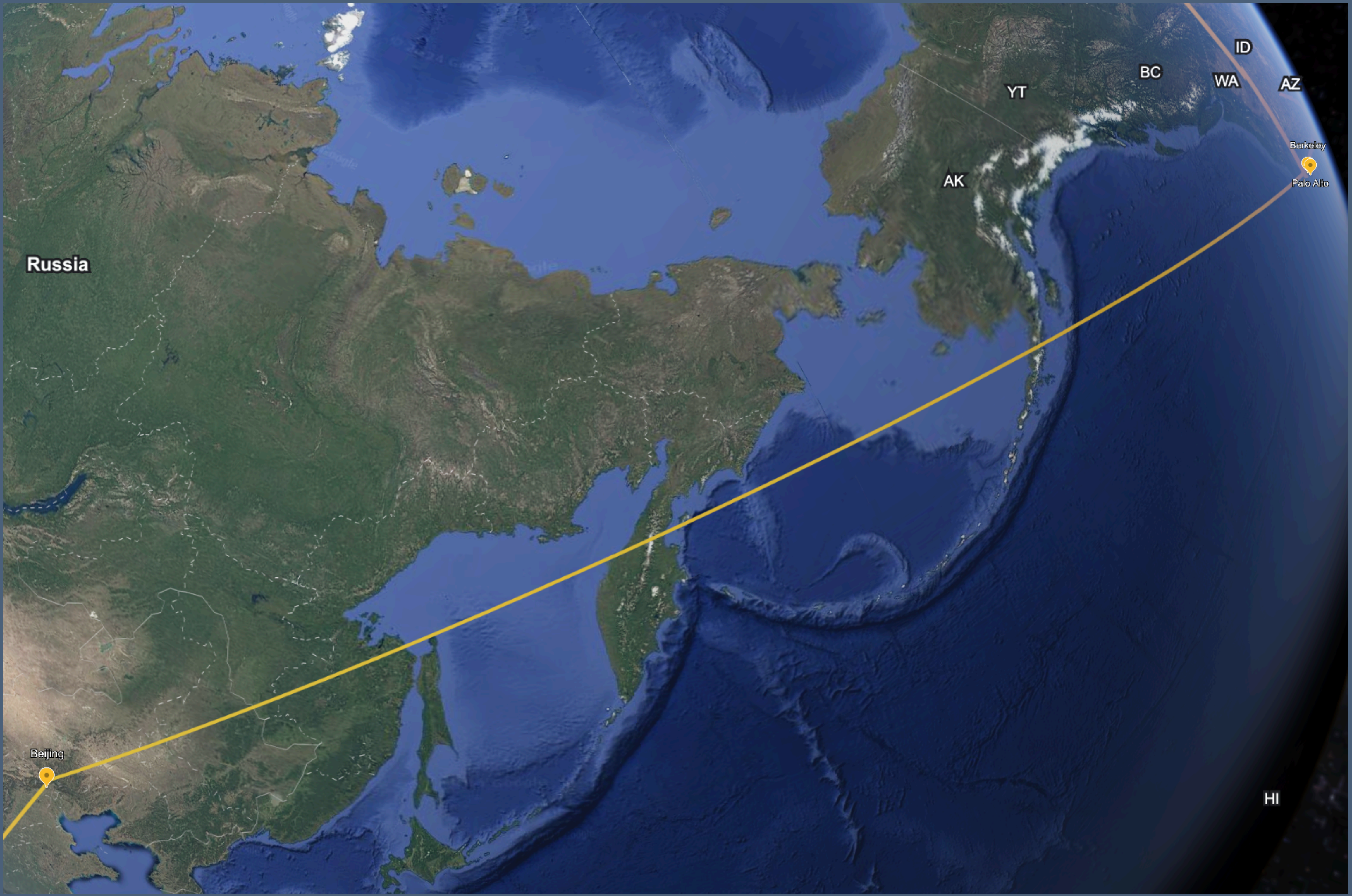
  $$\{M_\alpha : \alpha < \beta_n\} \subseteq \mathrm{NCR}_{G(n)}$$

  for a computable G.

Beijing

2006

- We needed a “non-acceleration” argument similar to the Turing jump for sequences of master codes.
- Since we cannot arithmetically define this sequence, we have to work with “pseudo”-codes and take the longest well-founded initial segment.
- Fortunately, a non-acceleration property holds for recognition of well-founded initial segments, too.
- This was the final puzzle piece!



Beijing - Palo Alto, 5897 miles

Palo Alto

2006

- THM (R. and Slaman):

  There exists a computable function $G(n)$ such that for every $n \in \omega$,

$$\mathrm{ZFC}_n^- \nvdash \text{"}\mathbf{NCR}_{G(n)} \text{ is countable."}$$

Years go past…

- Writing the paper: Developing the pseudocode machinery; need to work with ω-copies of Jensen's standard J-structures $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$.
- Let $G(N) = (N+2)(3c+6)$
- THM: Suppose $N \geq 0$, $\alpha < \beta_N$, and for some $n > 0$, $\rho_\alpha^n = 1$. Then the canonical copy $\langle X, M \rangle$ of the standard J-structure $\langle J_{\rho_\alpha^n}, A_\alpha^n \rangle$ is not $G(N)$-random with respect to any continuous measure.

Where to go from there…

- Two variations:
  - Other measures
  - Vary the notion of randomness

Hausdorff measures

- THM: Suppose X is such that for all n, $-\log \tilde{M}(X\lceil_n) \geq sn$, where $\tilde{M}$ is Levin's optimal continuous semimeasure. Then X is random with respect to a probability measure μ such that, for some γ,

$$(\forall \sigma)\ [\mu(\sigma) \leq \gamma 2^{-s|\sigma|}] \qquad (*)$$

- Frostman's Lemma:
  If A is a compact subset of 2^ω with $\mathscr{H}^s(A) > 0$, then there exists a probability measure μ such that $\mathrm{supp}(\mu) \subseteq A$, and such that there exists a constant γ such that (*) holds.

Point to Set

- In the proof of Frostman's Lemma, it is crucial that the μ-random real is a measure-theoretic "representative" of the support of μ.
- For Hausdorff dimension, this representation takes an even stronger form.
- THM (Cai & Hartmanis):
  For each $\alpha \in [0,1]$, $\dim_H\{X: \dim_H(X) = \alpha\} = \alpha$.
- This is an early version of the "point-to-set" principle (Lutz & Lutz).

Point to Set

- The Cai-Hartmanis phenomenon shows up in other contexts, too.
- THM (Jarnik; Besicovitch):

$$\dim_H\{x \in [0,1] : x \text{ has irrationality exponent } \alpha\} = \frac{2}{\alpha}$$

- The irrationality exponent of x is the supremum of all δ such that there exist infinitely many rational numbers p/q with $|x - p/q| < q^{-\delta}$.
- With a suitable rephrasing (introducing *Diophantine complexity*), this can be rewritten as an exact analogue of the Cai-Hartmanis result.

Point to Set

- Working hypothesis:
  ML-randomness (via Kolmogorov complexity) and Diophantine randomness are just two of a whole family of randomness notions exhibiting point-to-set behavior in the sense of Cai-Hartmanis, with ML-randomness being an extremal point.

- I have been able to confirm this for two kinds of complexity notions:
  - normal compressors   (introduced by Cilibrasi & Vitanyi)
  - abstract complexity measures   (introduced by Cotner & R.)

Fourier measures

- One of the most intriguing questions coming out of this line of investigation is:
"*Which reals are random with respect to a Fourier measure?*"

- A Borel measure μ on $\mathbb{R}$ is an α-Fourier measure if there exists a constant c such that for all $x \in \mathbb{R}$,

$$|\widehat{\mu}(x)| \leq c\,|x|^{-\alpha/2},$$

where $\widehat{\mu}$ is the *Fourier-Stieltjes transform* of μ.

Fourier measures

- The *Fourier dimension* of $A \subseteq \mathbb{R}$ is defined as

$$\dim_F A = \sup\{\alpha \colon \exists \alpha\text{-Fourier } \mu, \mu(A) = 1\}.$$

- It always holds that $\dim_F \leq \dim_H$, but they can be drastically different.
- For example, $\dim_F(\text{middle-third Cantor set}) = 0$.
- Sets for which $\dim_F = \dim_H$ are called *Salem sets*.

Fourier measures

- Kaufman showed that the Jarnik-Besicovitch fractal is a Salem set.
- THM (Slaman): The Cai-Hartmanis fractal is a Salem set.
- Conjecture:
  Every Cai-Hartmanis set for a reasonably strong complexity measure is a Salem set.