Math 566 - Homework 3
Due Wednesday February 7, 2024

1. Let R and S be rings, and let $f\colon R \to S$ be a ring homomorphism. Prove that if Q is a completely prime ideal of S that does not contain $f(R)$, then $f^{-1}(Q)$ is a completely prime ideal of R that contains $\ker(f)$.

2. Let R_1, $R_2, \ldots, R_n$ be rings with unity, and let I be an ideal of $R_1 \times \cdots \times R_n$. Prove that there exist ideals $J_i \triangleleft R_i$, $i = 1, \ldots, n$, such that $I = J_1 \times \cdots \times J_n$.

3. Let R be a ring, not necessarily commutative, and let $n \geq 1$. Then $M_n(R)$, the group of $n \times n$ matrices with coefficients in R, is a ring with the usual matrix multiplication. (You may take this for granted). Let J be a two-sided ideal of R. Prove that $M_n(J)$ is an ideal of $M_n(R)$.

4. Let R be a ring with unity, and let $S = M_n(R)$. Let J be a two-sided ideal of S. We will prove that $J = M_n(I)$ for some two-sided ideal I of R.

   (i) Let E_{rs} be the matrix that has 1_R in the (r, s) entry and 0s elsewhere. Show that $E_{rs}A$ is the matrix that has the sth row of A in the rth row, and zeros elsewhere. Give a similar description of AE_{rs} and prove that description holds.

   (ii Let I be the subset of all elements of R that appear as an entry of some element of J. Show that I is an ideal of R.

   (iii) Show that $a \in I$ if and only if there exists a matrix M in J such that a is the $(1, 1)$ entry of M, and all other entries of M are 0.

   (iv) Prove that $J = M_n(I)$.

5. Show that if R is a division ring, $n \geq 1$, and $S = M_n(R)$, then the zero ideal of S is a prime ideal. Show that if $n > 1$, then the zero ideal is not completely prime.

6. Let $R = 2\mathbb{Z}$, the ring of even integers. Show that $4\mathbb{Z}$ is a maximal ideal of R that is not a prime ideal, and show that $6\mathbb{Z}$ is both maximal and prime in R.

7. Let R be a ring, not necessarily commutative, not necessarily with unity. Let $f, g\colon \mathbb{Q} \to R$ be ring homomorphisms. Prove that if $f(n) = g(n)$ for all $n \in \mathbb{Z}$, then $f = g$.

8. Let R be a ring, not necessarily commutative, not necessarily with unity. Prove that the following are equivalent:

   (a) Every left ideal of R is finitely generated: if I is a left ideal of R, then there exist $a_1, \ldots, a_n \in I$ such that $I = (a_1, \ldots, a_n)$.

   (b) R satisfies ACC (the *Ascending Chain Condition*) on left ideals: that is, if we have $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ an ascending chain of left ideals of R, then there exists n such that $I_n = I_{n+j}$ for all $j \geq 0$.

   (c) Every nonempty collection $\mathcal{S}$ of left ideals of R has maximal elements: if $\mathcal{S}$ is a nonempty collection of left ideals of R, then there exists a left ideal $M \in \mathcal{S}$ such that for all left ideals $I \in \mathcal{S}$, if $M \subseteq I$ then $M = I$.