Name (PRINT): Answer Key

Student ID: ____________________

Math 1A, Final
Summer 2007
Instructor: Lynn Scow

Time: 1 hr 50 min

Instructions:

1. No calculators are allowed. A one-sided sheet of notes is permitted.
2. **Show all steps clearly**. Simply writing the solution without showing work will not receive full credit. For steps that are not the direct application of a rule, it is required that you further explain your answer in words.
3. Give only one answer per problem. Do not give two answers, or else only the first answer will be graded.
4. Good luck.

((Do not write under this line))

1	2	3	4	5	6	Total
15	15	20	20	15	15	100

1. (15 pts) Let $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x < 0 \\ \sqrt{x}, & \text{if } x \geq 0 \end{cases}$. Is f continuous on $\mathbb{R}$? Argue carefully for your conclusion.

f is not continuous on $\mathbb{R}$ because it is not continuous at the point $a=0$.

For f to be continuous at $a=0$

$\lim_{x \to a} f(x)$ must exist and $= f(a)$, however the limit DNE since $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} \sqrt{x} = \lim_{x \to 0^+} \sqrt{x} = 0$$

$$\lim_{x \to a^-} f(x) = \lim_{x \to a} \frac{\sin x}{x} = \lim_{x \to 0^-} \frac{\sin x}{x} = 1$$

$0 \neq 1$, thus $\lim_{x \to 0} f(x)$ DNE and $f(x)$ is not continuous at $a=0$.

2. (15 pts) Assume that $f(x)$ is differentiable and $f'(x)$ is continuous. Suppose that $f'(x) \leq 3$ for all $x \geq 15$. Prove that $f(17) \leq f(15) + 6$. [You may use any methods from the course.]

Since $f(x)$ is differentiable, it is continuous

so by the Mean Value Theorem there is a

c in $(15, 17)$ such that

$$\frac{f(17)-f(15)}{2} = \frac{f(17)-f(15)}{17-15} = f'(c)$$

since $c > 15$, $f'(c) \leq 3$

Thus,

$$\frac{f(17)-f(15)}{2} \leq 3$$

So,

$$f(17) - f(15) \leq 6$$

$$\Leftrightarrow \quad f(17) \leq f(15) + 6.$$

3. Evaluate the following limits.

(a) (10 pts) $\lim_{x\to 0+} x^{\sin x}$ $= y$

$$\ln\left(\lim_{x\to 0^+} x^{\sin x}\right) = \ln y$$

$$= \lim_{x\to 0^+} \ln x^{\sin x}$$

$$= \lim_{x\to 0^+} \sin x \cdot \ln x \qquad \text{" } 0\cdot -\infty \text{ "} \quad \text{indet.}$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} \qquad \text{" } \frac{-\infty}{\infty} \text{ "}$$

$$\overset{LH}{=} \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-\cos x}{\sin^2 x}\right)}$$

$$= \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \frac{-\sin x}{\cos x}$$

$$= \underbrace{\lim_{x\to 0^+} \frac{\sin x}{x}}_{1} \cdot \underbrace{\lim_{x\to 0^+} \frac{-\sin x}{\cos x}}_{0}$$

$$= 0$$

Thus $y = e^{\ln y} = e^0 = 1.$

(3. cont'd)

(b) (10 pts) $\lim_{x\to\infty} \dfrac{2x + \sin x}{\sqrt{x} + x + \cos x}$ $\dfrac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$

$$= \lim_{x\to\infty} \frac{2 + \frac{\sin x}{x}}{\frac{1}{\sqrt{x}} + 1 + \frac{\cos x}{x}} = \frac{2+0}{0+1+0} = \boxed{2}.$$

$\frac{\sin x}{x} \to 0$, $\frac{1}{\sqrt{x}} \to 0$, $\frac{\cos x}{x} \to 0$

$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$ Because $\lim_{x\to\infty} \sqrt{x} = \infty$

$\lim_{x\to\infty} 1 = 1$, a constant

$\lim_{x\to\infty} \frac{\sin x}{x} = 0$ Since $-1 \le \sin x \le 1$ for all x, thus

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

and since $\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to\infty} \frac{-1}{x} = 0$

By Squeeze Theorem

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

$\lim_{x\to\infty} \frac{\cos x}{x} = 0$ By similar argument since also

$-1 \le \cos x \le 1$ for all x.

4. Evaluate the following integrals.

(a) (10 pts) $\displaystyle\int_0^1 \frac{2+t}{1+t^2}\,dt$

$$= \int_0^1 \frac{2}{1+t^2}\,dt + \int_0^1 \frac{t}{1+t^2}\,dt$$

$u = 1+t^2$
$du = 2t\,dt$

$$= 2\arctan t\,\Big]_0^1 + \int_1^2 \frac{1}{2}\cdot\frac{1}{u}\,du \, .$$

$u(1) = 2$
$u(0) = 1$

$$= 2\underbrace{\arctan 1}_{\frac{\pi}{4}} - 2\underbrace{\arctan 0}_{0} + \frac{\ln|u|}{2}\Big]_1^2$$

$$= 2\left(\frac{\pi}{4}\right) - 0 + \frac{\ln(2)}{2} - \frac{\cancel{\ln(1)}^{\,0}}{2}$$

$$= \frac{\pi + \ln(2)}{2} \, .$$

(4. cont'd)

(b) (10 pts) $\displaystyle\int_1^{e^{2\pi}} \left|\frac{\sin(\ln x)}{x}\right| dx$

[Hint: first determine where the function $y = \frac{\sin(\ln x)}{x}$ is positive or negative on the interval $[1, e^{2\pi}]$.]

on $[0, 2\pi]$

$\sin t < 0$ if $\pi < t < 2\pi$

$\sin(\ln x) < 0$ if $\pi < \ln x < 2\pi \iff e^{\pi} < x < e^{2\pi}$

$\sin t > 0$ if $0 < t < \pi$.

$\sin(\ln x) > 0$ if $0 < \ln x < \pi \iff 1 < x < e^{\pi}$

Since $x > 0$, dividing by x does not change sign.

So $\displaystyle\int_1^{e^{2\pi}} \left|\frac{\sin(\ln x)}{x}\right| dx$

$$= \int_1^{e^{\pi}} \frac{\sin(\ln x)}{x}\,dx + \int_{e^{\pi}}^{e^{2\pi}} -\frac{\sin(\ln x)}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \int_0^{\pi} \sin(u)\cdot du - \int_{\pi}^{2\pi} \sin(u)\,du$$

$u(e^{2\pi}) = 2\pi$

$u(e^{\pi}) = \pi$

$u(1) = 0$

$$= -\cos u\Big]_0^{\pi} + \cos u\Big]_{\pi}^{2\pi}$$

$$= -\cos\pi + \cos 0 + \cos 2\pi - \cos\pi = -(-1) + 1 + 1 - (-1)$$

$$= \boxed{4}.$$

5. Let $g(x) = \int_0^{\ln x} \arctan t \, dt$ on the domain $x > 0$.

(a) (8 pts) Find $g'(x)$ and justify your answer.

$$F(x) = \int_0^x \arctan t \, dt$$

$$h(x) = \ln x$$

$$g(x) = F(h(x)) \longrightarrow g'(x) = F'(h(x)) \cdot h'(x)$$

by Chain Role.

$$F'(x) = \arctan x \quad \text{by FTC 1.}$$

So $$g'(x) = \frac{\arctan(\ln x)}{x}$$

(5. cont'd)

(b) (7 pts) Find the intervals of increase/decrease for $g(x)$ on its domain.

$$g'(x) = 0 = \frac{\arctan \ln x}{x} \iff$$

$$0 = \arctan(\ln x) \quad (\text{since } x > 0)$$

$$\iff 0 = \tan 0 = \ln x$$

$$\iff x = 1, \text{ critical point.}$$

$$\begin{array}{ccccc} & - & & + & \\ \hline 0 & & 1 & & \to g' \end{array}$$

if $0 < x < 1$, $\ln x < 0$

so $\arctan(\ln x) < 0$

so $\dfrac{\arctan(\ln x)}{x} < 0$

If $x > 1$, $\ln x > 0$

So $\arctan \ln x > 0$

so $\dfrac{\arctan \ln x}{x} > 0$

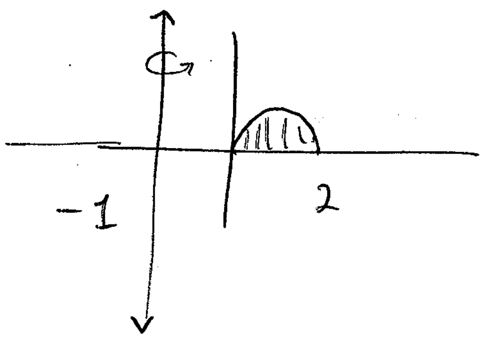
Thus by First Derivative Test,

$g(x)$ is increasing on $(1, \infty)$

decreasing on $(0, 1)$.

6. (15 pts) Find the volume generated by rotating the region bounded by $y = 2x - x^2$ and $y = 0$ about the axis $x = -1$.

$x(2-x)=0 \iff x=0,2$



use shells. For $1 \le x \le 2$

radius $= x+1$

height $= 2x-x^2$

~~area~~ volume $= \displaystyle\int_0^2 2\pi (x+1)(2x-x^2)\,dx$

$= 2\pi \displaystyle\int_0^2 x^2+2x-x^3\,dx$

$= 2\pi \left[\frac{x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^4}{4}\right]\Big]_0^2$

$= 2\pi\left[\frac{8}{3}+4-4\right] = \boxed{\frac{16\pi}{3}}$