

Name (PRINT): Answer Key

Student ID: _____

Math 1A, Midterm 2

Summer 2007

Instructor: Lynn Scow

Time: 50 minutes

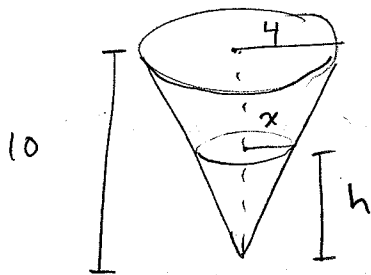
Instructions:

1. No calculators are allowed. A one-sided sheet of notes is permitted.
2. **Show all steps clearly.** Simply writing the solution without showing work will not receive full credit. For steps that are not the direct application of a rule, it is required that you further explain your answer in words.
3. Give only one answer per problem. Do not give two answers, or else only the first answer will be graded.
4. Good luck.

((Do not write under this line)) _____

1	20
2	20
3	20
4	20
Total:	80/80

1. (20 pts) A water tank has the shape of an inverted circular cone with base radius 4 m and height 10 m. If water is being pumped into the tank at a rate of $8 \text{ m}^3/\text{s}$, find the rate at which the water level is rising when the water is 5 m deep. [Recall: the volume of a cone of height h and radius r is $\frac{1}{3}\pi r^2 h$.]



$$\frac{x}{4} = \frac{h}{10}$$

$$\rightarrow x = \frac{4}{10}h = \frac{2}{5}h$$

$$\text{Let } t_1 : h(t_1) = 5$$

$$8 = \left. \frac{dV}{dt} \right|_{t=t_1} = \frac{4\pi}{25} (5)^2 \left. \frac{dh}{dt} \right|_{t=t_1}$$

$$\boxed{\frac{2}{\pi} \text{ m/s}} = \frac{8}{4\pi} = \left. \frac{dh}{dt} \right|_{t=t_1}$$

$$\begin{aligned} V &= \frac{1}{3} \pi x^2 h \\ &= \frac{1}{3} \pi \left(\frac{2}{5}h\right)^2 h \\ &= \frac{4\pi}{75} h^3 \end{aligned}$$

$$\frac{dV}{dt} = \frac{3 \cdot 4\pi}{75} h^2 \frac{dh}{dt}$$

$$= \frac{4\pi}{25} h^2 \frac{dh}{dt}$$

2. (20 pts) Let c be any real constant. Show that $f(x) = x^4 + 2x^2 + c$ has at most two real roots.

Suppose, for contradiction, that f has at least three real roots, call them x, y, z .

$$\text{Then } f(x) = 0, f(y) = 0, f(z) = 0.$$

Thus $f(x) = f(y)$, and since f is differentiable on (x, y) and continuous on $[x, y]$ (as a polynomial, it is differentiable everywhere), by Rolle's Theorem there is a c_1 in (x, y)

$$\text{such that } f'(c_1) = 0$$

Similarly, since $f(y) = f(z)$ and f is differentiable everywhere, by Rolle's Theorem there is c_2 in (y, z)

$$\text{such that } f'(c_2) = 0$$

$c_1 \neq c_2$, so f' must have at least two real roots.

$$f'(x) = 4x^3 + 4x = 4x(x^2 + 1) = 0 \iff x = 0 \text{ or } x^2 + 1 = 0 \text{ (impossible)}$$

We see by inspection that f' has only one real root, contradiction.

Thus f has no more than two real roots.

3. Find the following three limits.

(a) (6 pts) $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3 + x^4}$ " $\frac{0}{0}$ "

$$\stackrel{\text{LH}}{=} \lim_{x \rightarrow 0} \frac{\cos x - 1}{3x^2 + 4x^3} \quad \frac{0}{0}$$

$$\stackrel{\text{LH}}{=} \lim_{x \rightarrow 0} \frac{-\sin x}{6x + 12x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{-1}{6 + 12x}$$

$$= 1 \cdot \frac{-1}{6 + 0} = \boxed{\frac{-1}{6}}$$

(b) (6 pts) $\lim_{x \rightarrow 2\pi^+} \frac{\ln(x - 2\pi)}{\sin x}$

" $\frac{-\infty}{0}$ "

may not apply
L'Hospital's Rule

$$\lim_{x \rightarrow 2\pi^+} \ln(x - 2\pi) = -\infty$$

$$\lim_{x \rightarrow 2\pi^+} \frac{1}{\sin x} = +\infty \quad (\text{since } \sin x \rightarrow 0^+)$$

Thus, as x tends to 2π from the right

$\ln(x - 2\pi) \cdot \frac{1}{\sin x}$ becomes an arbitrarily large

negative number, so $\lim_{x \rightarrow 2\pi^+} \frac{\ln(x - 2\pi)}{\sin x} = -\infty$.

(3. cont'd)

$$(c) (8 \text{ pts}) \lim_{x \rightarrow 0^+} (\cos 2x)^{3/x} = y \quad \text{"} 1^\infty \text{"}$$

$$\ln \left(\lim_{x \rightarrow 0^+} (\cos 2x)^{3/x} \right) = \ln y$$

$$\lim_{x \rightarrow 0} \ln (\cos 2x)^{3/x} = \ln y$$

$$= \lim_{x \rightarrow 0} \frac{3}{x} \cdot \ln \cos 2x = \lim_{x \rightarrow 0} \frac{3 \cdot \ln(\cos 2x)}{x} \quad \text{"} \frac{0}{0} \text{"}$$

$$\stackrel{\text{LH}}{=} \lim_{x \rightarrow 0} \frac{3}{\cos 2x} \cdot \frac{-\sin 2x \cdot 2}{1}$$

$$= \lim_{x \rightarrow 0} -6 \cdot \tan 2x = 0$$

Thus $y = e^{\ln y} = e^0 = \boxed{1}$

4. (20 pts) Find all points on the graph of $f(x) = \sqrt{x^3 + 1}$ restricted to the domain $[-1, 1]$ that are the minimum distance from the origin and also those points on the graph of f restricted to the same domain that are the maximum distance from the origin.

$$d = \sqrt{(x-0)^2 + (y-0)^2} \quad \rightarrow D = d^2 = x^2 + y^2$$

$$= x^2 + (\sqrt{x^3 + 1})^2$$

$$= x^2 + x^3 + 1$$

$$D'(x) = 2x + 3x^2 = x(2 + 3x) = 0$$

$$\leftrightarrow x = 0, \quad x = -\frac{2}{3} \quad \text{critical pts of}$$

x	D(x)	
-1	1	← abs min
$-\frac{2}{3}$	$\frac{31}{27}$	
0	1	← abs min
1	3	← abs max $(3 = \frac{81}{27})$

since $D(x)$ has an absolute max/min on $[a, b]$ at x if and only if $d(x)$ does.

So the points a minimum distance from $(0, 0)$ on $f(x)$ are: $(-1, 0), (0, 1)$

and the point a maximum distance from $(0, 0)$ is: $(1, \sqrt{2})$