Section 11.11: Binomial Series examples

Example 1. Expand $f(x) = \frac{1}{(1+x)^4}$ as a binomial series.

Solution: Write

$$f(x) = (1+x)^{-4}$$

We know that $(1+x)^k$ is represented as $\Sigma_{n=0}^{\infty}\binom{k}{n}x^n$ for $|x| < 1$ so by substitution

$$f(x) = \Sigma_{n=0}^{\infty}\binom{-4}{n}x^n$$

Using the definition of $\binom{k}{n}$ we simplify to get

$$f(x) = 1 + \Sigma_{n=1}^{\infty}\frac{-4 \cdot -4-1 \cdot -4-2 \cdot \ldots \cdot -4-(n-1)}{n!}x^n$$

(we pulled out the 1 because $-4-(n-1)$ gives us the first term in the expanded product when we set $n = 1$ but setting $n = 0$ gives us something we don't want.)

Let's rewrite:

$$f(x) = 1 + \Sigma_{n=1}^{\infty}\frac{-4 \cdot -5 \cdot -6 \cdot \ldots \cdot (-3-n)}{n!}x^n$$

Consider that since there are n terms in the product (the size of $\{0, 1, 2, \ldots n-1\}$) then if n is even, the product is positive, and if n is odd, the product is negative. Let's represent that with $(-1)^n$

$$f(x) = \Sigma_{n=1}^{\infty}(-1)^n\frac{4 \cdot 5 \cdot 6 \cdot \ldots \cdot (n+3)}{n!}x^n$$

Now we notice that the product is $(n+3)!$ divided by $3!$:

$$f(x) = \Sigma_{n=1}^{\infty}(-1)^n\frac{(n+3)!}{3!n!}x^n$$

And we're done simplifying.

Example 2. Here's a slightly harder series. $f(x) = \sqrt[4]{1-8x}$.

Solution. There is no need to pull out the 8, just notice that $f(x) = (1 + (-8x))^{1/4}$ so that you are substituting $-8x$ for x in the original series (note that this changes the radius of convergence however, and you get $|x| < 1/8$.)

$$f(x) = \Sigma_{n=0}^{\infty} \binom{1/4}{n} (-1)^n 8^n x^n$$

Now we can try to simplify a bit. This case is a little harder because k is not an integer.

$$f(x) = 1 + \Sigma_{n=1}^{\infty} \frac{1/4 \cdot 1/4 - 1 \cdot 1/4 - 2 \cdot \ldots \cdot 1/4 - (n-1)}{n!} (-1)^n 8^n x^n$$

But we can multiply top and bottom by n copies of 4 (the number of factors) so as to only have integers in the numerator and denominator.

$$f(x) = 1 + \Sigma_{n=1}^{\infty} \frac{1 \cdot 1 - 4 \cdot 1 - 2(4) \cdot \ldots \cdot 1 - (n-1)4}{4^n n!} (-1)^n 8^n x^n$$

Immediately we can notice that 8^n over 4^n is 2^n:

$$f(x) = 1 + \Sigma_{n=1}^{\infty} \frac{1 \cdot 1 - 4 \cdot 1 - 2(4) \cdot \ldots \cdot 1 - (n-1)4}{n!} (-1)^n 2^n x^n$$

Now let's simplify the differences:

$$f(x) = 1 + \Sigma_{n=1}^{\infty} \frac{1 \cdot -3 \cdot -7 \cdot \ldots \cdot 5 - 4n}{n!} (-1)^n 2^n x^n$$

In this case the sign of the product is positive if there are an even number of negative terms, which happens if n is odd. And the product is negative if there are an odd number of negative terms, which happens if n is even. Thus the sign may be represented by $(-1)^{n+1}$. However we'd better take out one more term because $4n - 5$ gives -1 at $n = 1$ when it should give 1:

$$f(x) = 1 - 2x + \Sigma_{n=2}^{\infty} (-1)^{n+1} \frac{3 \cdot 7 \cdot \ldots \cdot 4n - 5}{n!} (-1)^n 2^n x^n$$

Let's simplify the sign again, $(-1)^{2n+1}$ is just -1:

$$f(x) = 1 - 2x - \Sigma_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \ldots \cdot 4n - 5}{n!} 2^n x^n$$

And we're done.