Answers.

Name (PRINT): ____________________

Student ID: ________________________

Math 16B, Final
Summer 2009
Instructor: Lynn Scow

Time: 1 hr 50 min

Instructions:

1. No calculators are allowed.
2. **Show all steps clearly**. Show work for every problem.
3. Give only one answer per problem. Do not give two answers, or else only the first answer will be graded.
4. Solve all values of sine and cosine, but no need to simplify purely algebraic expressions.

1	2	3	4	5	6	7	Total

1. (20 points) Suppose that x units of labor and y units of capital can produce $f(x, y) = 120x^{3/4}y^{1/4}$ units of product at your factory. Suppose each unit of labor costs \$30 and each unit of capital costs \$10.

   How many units of labor and how many units of capital should be utilized in order to maximize production assuming you have \$400 available to spend?

$$30x + 10y = 400$$

$$F(x,y,\lambda) = 120x^{3/4}y^{1/4} + \lambda(30x + 10y - 400)$$

$$\frac{\partial F}{\partial x} = \frac{3}{\cancel{4}}\overset{30}{\cancel{120}}x^{-1/4}y^{1/4} + 30\lambda = 0$$

$$\frac{\partial F}{\partial y} = \frac{1}{\cancel{4}}\overset{30}{\cancel{120}}y^{-3/4}x^{3/4} + 10\lambda = 0$$

$$\frac{\overset{3}{\cancel{90}}x^{-1/4}y^{1/4}}{\cancel{30}} = -\lambda = \frac{\overset{3}{\cancel{30}}y^{-3/4}x^{3/4}}{\cancel{10}}$$

$$3x^{-1/4}y^{1/4} = 3y^{-3/4}x^{3/4}$$

$$3y^{1/4} = 3y^{-3/4}x$$

$$3y = 3x$$

$$y = x$$

$$\frac{\partial F}{\partial \lambda} = 30x + 10y - 400 = 0$$

$$30x + 10x - 400 = 0$$

$$40x = 400$$

$$x = 10$$

$$y = 10$$

10 units of labor
10 units of capital

2. (over for part (b))

(a) (10 points) Find the indefinite integral: $\int x\sqrt{2-x}\,dx$

$u = x \qquad dv = \sqrt{2-x}$

$du = dx \qquad v = \frac{-2}{3}(2-x)^{3/2}$

$-\frac{2}{3}x(2-x)^{3/2} + \int \frac{2}{3}(2-x)^{3/2}\,dx$

$= \boxed{-\frac{2}{3}x\,(2-x)^{3/2} + \frac{2}{3}\cdot\frac{-2}{5}\cdot(2-x)^{5/2} + C}$

$\int \sqrt{2-x}\,dx = \frac{(2-x)^{3/2}}{3/2}(-1)$

$= -\frac{2}{3}(2-x)^{3/2}$

$\int (2-x)^{3/2}\,dx = \frac{(2-x)^{5/2}}{5/2}\cdot(-1)$

(b) (10 points) Find the definite integral: $\displaystyle\int_{2\pi^2/9}^{8\pi^2/9} -\left(\frac{\sin[\sqrt{2x}-\frac{\pi}{3}]}{\sqrt{2x}}\right) dx$

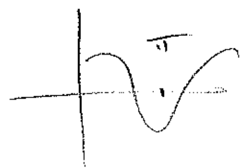
$$= \int_{\frac{\pi}{3}}^{\pi} -\sin u \ du$$

$$u = \sqrt{2x} - \frac{\pi}{3}$$

$$du = \cancel{\frac{1}{2}} \frac{\cancel{2}}{\sqrt{2x}} dx$$

$$= \frac{dx}{\sqrt{2x}}$$

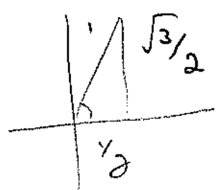
$$= \cos u \Big]_{\pi/3}^{\pi}$$



$$= \cos\pi - \cos\frac{\pi}{3}$$

$$= (-1) - \left(\frac{1}{2}\right)$$

$$= \boxed{\frac{-3}{2}}$$



x	$u = \sqrt{2x} - \frac{\pi}{3}$
$\frac{8\pi^2}{9}$	$\sqrt{\frac{16\pi^2}{9}} - \frac{\pi}{3} = \frac{4\pi}{3} - \frac{\pi}{3} = \pi$
$2\pi^2/9$	$\sqrt{\frac{4\pi^2}{9}} - \frac{\pi}{3} = \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$

3. (20 points) Solve the initial value problem:

$$\frac{dy}{dt} = \frac{t+1}{ty^2}, \quad y(1) = 2 \qquad \text{for } t > 0$$

$$\int y^2\, dy = \frac{t+1}{t}\, dt = \int 1 + \frac{1}{t}\, dt$$

$$\frac{y^3}{3} = t + \ln t + C \qquad (t > 0)$$

$$y^3 = 3t + 3\ln t + C$$

$$y = \sqrt[3]{3t + 3\ln t + C}$$

$$y(1) = (3 + 0 + c)^{1/3} = 2$$

$$3 + C = 2^3 = 8$$

$$C = 5$$

$$\boxed{y(t) = \sqrt[3]{3t + 3\ln t + 5}}$$

4. (10 points) Suppose you have a bank account that earns 5% interest compounded continuously and you make deposits into the account at a rate of \$1,000 a year. Suppose your elderly parent makes withdrawals from your account at a rate of \$5,000. You may assume all deposits and withdrawals are made continuously throughout the year.

If you start with an initial balance of \$70,000, does the account ever run out of money?

Please answer this question using **one** of two options:

(option a) Set up and solve a differential equation for $y(t)$=the amount of money in the account at time t. Reason using limits or algebra.

(option b) Set up a yz-axis and a ty-axis and make a full sketch of the solution $y(t)$ to this initial value problem. Reason using the theory of differential equations.

$\frac{4000}{.05} = 80000$

(a)

$$\frac{dy}{dt} = .05y + 1000 - 5000$$

$$= .05y - 4000$$

$$y' - .05y = -4000$$

$$y(t) = \frac{\int e^{-.05t}(-4000)\,dt}{e^{-.05t}}$$

$$= \frac{\frac{-4000}{-.05} e^{-.05t} + C}{e^{-.05t}}$$

$$= 80{,}000 + Ce^{.05t}$$

$$y(0) = 80{,}000 + C = 70{,}000$$

$$C = -10{,}000$$

$$y(t) = 80{,}000 - 10{,}000e^{.05t}$$

$\lim_{t\to\infty} y(t) = -\infty$ so

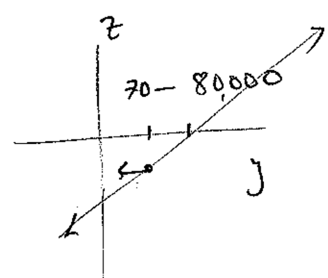
$y(t) = 0$ for some $t > 0$.

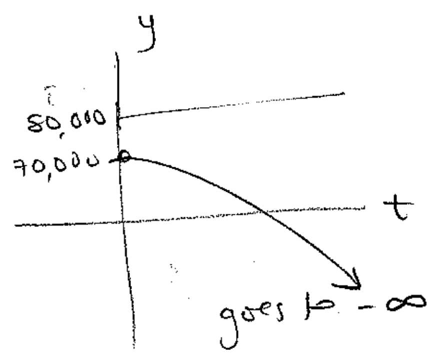
yes acct runs out.

— or —

< Solve for t in $y(t) = 0$. >

(b) $\frac{dy}{dt} = 0 \longrightarrow y = \frac{4000}{.05} = 80{,}000$



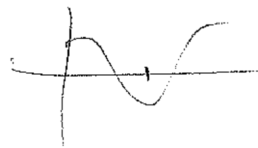


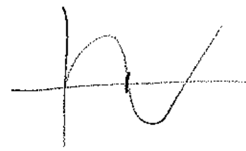
crosses t-axis

$\Rightarrow$ acct runs out.

5. (20 points) Use the fourth Taylor polynomial of $\cos x$ at $x = \pi$ to approximate $\cos 3$.

		at $x = \pi$
$n = 0$	$f = \cos x$	-1
1	$f' = -\sin x$	0
2	$f'' = -\cos x$	1
3	$f''' = \sin x$	0
4	$f^{(4)} = \cos x$	-1


cos x


sin x

$$P_4(x) = -1 + \frac{1}{2!}(x-\pi)^2 + \frac{(-1)}{4!}(x-\pi)^4$$

$$P_4(x) \approx f(x)$$

$$P_4(3) \approx f(3) = \cos 3$$

$$\boxed{P_4(3) = -1 + \frac{(3-\pi)^2}{2} - \frac{(3-\pi)^4}{24}}$$

6. (10 points) Use the integral test to determine convergence or divergence of $\Sigma_{k=1}^{\infty}\dfrac{k+3}{(k^2+6k+2)^2}$.

You may assume that the integral test applies (i.e., that the associated function is continuous, positive and decreasing.)

$$\int_1^{\infty} \frac{x+3}{(x^2+6x+2)^2}\,dx$$

$u = x^2 + 6x + 2$

$du = 2x + 6\ dx$

$\frac{1}{2} du = x + 3\ dx$

x	u
b	b^2+6b+2
1	$1+6+2=9$

$$= \lim_{b\to\infty} \int_9^{b^2+6b+2} \frac{\frac{1}{2}du}{u^2}$$

$$= \lim_{b\to\infty} \frac{-1}{2u}\Big]_9^{b^2+6b+2}$$

$$= \lim_{b\to\infty} \cancel{\frac{-1}{2(b^2+6b+2)}}^{\,0} + \frac{1}{18} = \frac{1}{18} \quad \text{converges}$$

thus, series converges

7. (over for part (b))

(a) (10 points) Suppose your potential profit (in tens of thousands of dollars) from an investment in Google is a random variable X with density function $f(x) = \dfrac{2}{x^3}$ on $[1, \infty)$.

On the other hand, you are likely to make a profit of \$50,000 from an investment in Yahoo, but there is a 30% chance you could make only \$40,000 and a 10% chance you could make as low as \$30,000 from this investment.

Y

If you commit yourself to a long-term investment in only one company, which company should it be?

$$E(x) = \int_1^\infty x\left(\frac{2}{x^3}\right)dx = \lim_{b\to\infty}\int_1^b \frac{2}{x^2}\,dx = \lim_{b\to\infty} \left.\frac{-2}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \cancel{\frac{-2}{b}}^{\,0} + \frac{2}{1} = \boxed{2}$$

table for Y: $E(Y) = .6(50{,}000) + .3(40{,}000) + .1(30{,}000)$

profit	50—	40—	30—
prob	.6	.30	.10

$= 30{,}000 + 12{,}000 + 3{,}000$

$= \boxed{45{,}000}$

X yields avg of \$20,000 profit

Y yields avg of \$45,000 profit

invest in Yahoo!

(b) (10 points) Snappy Solutions is a Texas-based company that makes millions selling calculators to high school students. Let X be an exponential random variable representing the lifetime of a Snappy Solutions calculator. Suppose 80% of Snappy Solutions calculators fail during the first 4 years. What is the average lifetime of a Snappy Solutions calculator? $= E(X)$

You may make use of any of the following approximations:

$\ln .1 = -2.3$

$\ln .2 = -1.6$

$\ln .3 = -1.2$

$\ln .4 = -.92$

$f(x) = ke^{-kx} \quad E(x) = \frac{1}{k}$

$Pr(x \leq 4) = .8$ 80% have a lifetime ≤ 4 yrs

$$\int_0^4 ke^{-kx} = .8$$

$$\frac{ke^{-kx}}{-k}\Big]_0^4 = -e^{-kx}\Big]_0^4 = -e^{-4k} + \cancel{e^{0}}\,1$$

$$= 1 - \frac{1}{e^{4k}} = .8$$

So $-\frac{1}{e^{4k}} = .8 - 1 = -.2$

$e^{-4k} = .2$

$-4k = \ln .2$

$k = \frac{\ln .2}{-4} = \frac{-1.6}{-4} = .4$

so $E(x) = \frac{10}{4} = 2.5$

$\boxed{\text{avg lifetime } 2.5 \text{ yrs}}$