

# Matrix Valued Orthogonal Polynomials on the Unit Circle: Some Extensions of the Classical Theory

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*Abstract.* In the work presented below the classical subject of orthogonal polynomials on the unit circle is discussed in the matrix setting. An explicit matrix representation of the matrix valued orthogonal polynomials in terms of the moments of the measure is presented. Classical recurrence relations are revisited using the matrix representation of the polynomials. The matrix expressions for the kernel polynomials and the Christoffel–Darboux formulas are presented for the first time.

## 1 Introduction

Since the fundamental work of Akhiezer [1], Szegő [32], and many others, orthogonal polynomials have been very extensively used in analyzing many problems of applied mathematics, such as numerical quadrature, the moment problem, rational and polynomial interpolation and approximation, and applications of these techniques in engineering problems. The development of special and important examples goes much further back; see for instance Lebedev [26]. Numerous applications of matrix valued orthogonal polynomials supported on the unit circle include the inversion of finite block Toeplitz matrices that appear in linear estimation theory, see [29]; application in time series analysis related to the frequency estimation of a stationary harmonic process, see [31]; the analysis of families of polynomials orthogonal with respect to scalar measure supported on equipotential curves in the complex plane, see [27] and many others.

Starting with the earlier work of M. G. Krein [24, 25] as well as more recent works [2, 6–11, 13, 15, 18, 19, 28, 31], there is a general theory of matrix valued orthogonal polynomials. Many of the important results of the theory of scalar valued orthogonal polynomials, such as Favard’s theorem and Markov’s theorem, have been adapted in the matrix setting, see [6–9, 13, 14, 19], and many more still need to be investigated in the new context of matrix valued orthogonal polynomials.

In this article we utilize the moments of the orthogonality measure to represent matrix valued orthogonal polynomials on the unit circle as certain Schur complements. The idea of representing matrix valued orthogonal polynomials using Schur complements has been discussed by several authors in various contexts. In particular, Schur complements representation has been used in block-algorithms in numerical

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linear algebra, see [4, 16, 21] and many more. The current methodology of using orthogonality measure and Schur complements representation described in this article allows one to obtain classical recursion relations in a simple way. Most importantly, matrix versions of the kernel polynomials and Christoffel–Darboux formulas are presented for the first time.

This paper is organized as follows. In Section 2 notations are introduced and the matrix analog of the determinant formula for the polynomials on the unit circle is presented. Section 3 concerns orthogonality of the polynomials introduced in Section 2. The recurrence relations in the matrix case are presented in Section 4. Section 5 concerns the matrix valued version of the kernel polynomials and the Christoffel–Darboux formulas.

## 2 Orthogonal Polynomials and the Moments of Measure

In [6, 19] the subject of matrix valued orthogonal polynomials on the unit circle was approached from the point of view of a minimization problem. Presented below is an explicit matrix expression for the scalar/matrix valued orthogonal polynomials on the unit circle in terms of the moments of the measure which is a natural extension of the classical determinant definition discussed in numerous books and articles, for example, see [5].

Given a measure  $\mu(d\theta) = W(\theta)d\theta$  with Hermitian weight function  $W(\theta) \in \mathbf{R}^{k \times k}$ ,  $k \geq 1$ , supported and integrable on  $[-\pi, \pi]$ , introduce the following:

- The  $n$ -th moment of the measure  $\mu(d\theta)$   $\mu_n \in \mathbf{C}^{k \times k}$ , where

$$\mu_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} \mu(d\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} W(\theta) d\theta; \quad n = 0, \pm 1, \pm 2 \dots$$

Note that  $\mu_{-n} = \mu_n^*$ . Throughout this section “\*” means transposition and complex conjugation.

- The matrices  $M_n^r$  and  $M_n^l$  in  $\mathbf{C}^{k(n+1) \times k(n+1)}$ , where  $I$  is  $k \times k$  identity matrix,  $x = e^{i\theta} \in \mathbf{C}$ ,  $\theta \in [-\pi, \pi]$  and  $n \geq 1$ ,

$$M_n^r = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & \mu_n \\ \mu_{-1} & \mu_0 & \cdots & \mu_{-n+2} & \mu_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_0 & \mu_1 \\ I & xI & \cdots & x^{n-1}I & x^n I \end{pmatrix},$$

$$M_n^l = \begin{pmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n+1} & I \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+2} & xI \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_{n-2} & \cdots & \mu_0 & x^{n-1}I \\ \mu_n & \mu_{n-1} & \cdots & \mu_1 & x^n I \end{pmatrix}.$$

- Toeplitz matrices  $T_n^r$  and  $T_n^l \in \mathbf{C}^{kn \times kn}$  for  $n \geq 1$

$$T_n^r = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_{-1} & \mu_0 & \cdots & \mu_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{-n+1} & \mu_{-n+2} & \cdots & \mu_0 \end{pmatrix} \text{ and } T_n^l = \begin{pmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n+1} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_{n-2} & \cdots & \mu_0 \end{pmatrix}.$$

- The vectors  $\nu_n$  and  $\xi_n$  for  $n \geq 1$ ,

$$\nu_n = \begin{pmatrix} \mu_n \\ \mu_{n-1} \\ \vdots \\ \mu_1 \end{pmatrix} \text{ and } \xi_n = \begin{pmatrix} \mu_{-n} \\ \mu_{-n+1} \\ \vdots \\ \mu_{-1} \end{pmatrix}.$$

- In the matrices

$$T_{n+1}^r = \begin{pmatrix} T_n^r & \nu_n \\ \nu_n^* & \mu_0 \end{pmatrix} \text{ and } T_{n+1}^l = \begin{pmatrix} T_n^l & \xi_n \\ \xi_n^* & \mu_0 \end{pmatrix}$$

denote the Schur complements of  $\mu_0$

$$(1) \quad S_n^r = \mu_0 - \nu_n^* T_n^{-r} \nu_n, \quad S_n^l = \mu_0 - \xi_n^* T_n^{-l} \xi_n, \text{ for } n \geq 1$$

with  $S_0^l = S_0^r = \mu_0$ . Here  $T_n^{-l}$  and  $T_n^{-r}$  denote  $(T_n^l)^{-1}$  and  $(T_n^r)^{-1}$  correspondingly.

Using the notations above we introduce the following definition:

**Definition 2.1 (Monic matrix valued polynomials on the unit circle)** Define two families of polynomials  $\{P_n^r(x)\}_{n=0}^\infty$  and  $\{P_n^l(x)\}_{n=0}^\infty$  as Schur complements of  $x^n I$  in the matrices  $M_{n+1}^r$  and  $M_{n+1}^l$  correspondingly, i.e.,

$$(2) \quad P_n^r(x) = x^n I - \begin{bmatrix} I & \cdots & x^{n-1} I \end{bmatrix} T_n^{-r} \begin{pmatrix} \mu_n \\ \mu_{n-1} \\ \vdots \\ \mu_1 \end{pmatrix}$$

and

$$(3) \quad P_n^l(x) = x^n I - \begin{bmatrix} \mu_n & \mu_{n-1} & \cdots & \mu_1 \end{bmatrix} T_n^{-l} \begin{pmatrix} I \\ xI \\ \vdots \\ x^{n-1} I \end{pmatrix},$$

with  $P_0^l(x) = P_0^r(x) = I$ , where “r” and “l” stand for the “right” and the “left” polynomials.

**Note 2.2** In the classical theory of scalar valued orthogonal polynomials on the unit circle (see [32]), monic polynomials are defined as

$$p_n^r(x) = p_n^l(x) = \frac{\det(M_n^r)}{\det(T_n^r)},$$

which is exactly what we obtain using definition (2) in the scalar case.

### 3 Orthogonality via the Moments of the Measure

The following proposition shows that families of monic polynomials  $\{P_n^r(x)\}_{n=0}^\infty$  and  $\{P_n^l(x)\}_{n=0}^\infty$  as defined in (2) and (3) form sets of monic orthogonal polynomials for any symmetric measure  $\mu(d\theta) = W(\theta)d\theta$ .

**Proposition 3.1** *Let  $\{P_n^r(x), P_n^l(x)\}_{n=0}^\infty$  be families of monic polynomials as defined in (2) and (3); let  $S_n^r$  and  $S_n^l$  be defined as in (1). Define “right” and “left” inner products on the unit circle as*

$$\begin{aligned}\langle P, Q \rangle_r &= \int P^*(e^{i\theta})W(e^{i\theta})Q(e^{i\theta})d\theta, \\ \langle P, Q \rangle_l &= \int P(e^{i\theta})W(e^{i\theta})Q^*(e^{i\theta})d\theta.\end{aligned}$$

Then for any  $k, j \geq 0$

$$\begin{aligned}\langle P_k^r, P_j^r \rangle_r &= \int_{-\pi}^{\pi} P_k^{r*}(e^{i\theta})W(\theta)P_j^r(e^{i\theta})d\theta = \delta_{kj}S_k^r, \\ \langle P_k^l, P_j^l \rangle_l &= \int_{-\pi}^{\pi} P_k^l(e^{i\theta})W(\theta)P_j^{l*}(e^{i\theta})d\theta = \delta_{kj}S_k^l.\end{aligned}$$

**Proof** Let us consider the right norm. Observe first that for any  $0 \leq m \leq n-1$

$$[\mu_{-m} \quad \mu_{-m+1} \quad \mu_{-m+2} \quad \cdots \quad \mu_{-m+n-1}] T_n^{-r} \nu_n = \mu_{-m+n}.$$

In order to prove that  $\langle P_m^r, P_n^r \rangle_r = 0$  for any  $m < n$  it is enough to show that  $P_n^r(e^{i\theta})$  is orthogonal to all  $e^{-im\theta}$  for  $0 \leq m \leq n-1$ , i.e.,

$$\begin{aligned}\int_{-\pi}^{\pi} e^{-im\theta}W(\theta)P_n^r(e^{i\theta})d\theta &= \int_{-\pi}^{\pi} e^{-im\theta}W(\theta)(e^{in\theta}I - [I \quad \cdots \quad e^{i(n-1)\theta}I])T_n^{-r}\nu_n d\theta \\ &= \mu_{n-m} - [\mu_{-m} \quad \mu_{-m+1} \quad \cdots \quad \mu_{-m+n-1}]T_n^{-r}\nu_n \\ &= \mu_{n-m} - \mu_{n-m} = 0.\end{aligned}$$

If  $m = n$ , then

$$\begin{aligned}\int P_n^{r*}(e^{i\theta})W(\theta)P_n^r(e^{i\theta})d\theta &= \mu_0 - [\mu_{-n} \quad \mu_{-n+1} \quad \cdots \quad \mu_{-1}]T_n^{-r}\nu_n \\ &= \mu_0 - \nu_n^*T_n^{-r}\nu_n = S_n^r.\end{aligned}$$

The statement for the left norm is proved similarly. ■

**Note 3.2** In the classical theory of scalar valued orthogonal polynomials on the unit circle (for example, see [32]),

$$\langle P_n^r, P_n^r \rangle_r = \frac{\det(T_{n+1})}{\det(T_n)},$$

which is identical to our formula applied for the scalar case.

## 4 The Recursion Relations

It is shown below that the right and the left orthogonal polynomials as defined in (2) and (3) obey the classical recurrence relations discussed in numerous articles, see for example [6, 19, 31] and many more.

**Proposition 4.1** *Let  $\{P_n^r(x)\}_{n=0}^\infty$  and  $\{P_n^l(x)\}_{n=0}^\infty$  be families of monic matrix valued orthogonal polynomials as defined in (2) and (3). Then they obey the following recursion relations:*

- (i)  $P_{n+1}^r(x) = xP_n^r(x) + \hat{P}_n^l(x)P_{n+1}^r(0);$
  - (ii)  $\hat{P}_{n+1}^r(x) = \hat{P}_n^r(x) + xP_{n+1}^{r*}(0)P_n^l(x);$
  - (iii)  $P_{n+1}^l(x) = xP_n^l(x) + P_{n+1}^l(0)\hat{P}_n^r(x);$
  - (iv)  $\hat{P}_{n+1}^l(x) = \hat{P}_n^l(x) + xP_n^r(x)P_{n+1}^{l*}(0);$
  - (v)  $P_{n+1}^r(x) = xP_n^r(x)(I - P_{n+1}^{l*}(0)P_{n+1}^r(0)) + \hat{P}_{n+1}^l(x)P_{n+1}^r(0);$
  - (vi)  $P_{n+1}^l(x) = x(I - P_{n+1}^l(0)P_{n+1}^{r*}(0))P_n^l(x) + P_{n+1}^l(0)\hat{P}_{n+1}^r(x);$
  - (vii)  $(I - P_{n+1}^{l*}(0)P_{n+1}^r(0)) = S_n^{-r}S_{n+1}^r;$
  - (viii)  $(I - P_{n+1}^l(0)P_{n+1}^{r*}(0)) = S_{n+1}^lS_n^{-l};$
  - (ix)  $S_n^lP_n^r(0) = P_n^l(0)S_n^r;$
- where  $\hat{P}_n^{r,l}(x) = x^n (P_n^{r,l}(x))^*$ .

**Proof** In order to prove the first recursion relation let us partition matrices  $T_{n+1}^r$ ,  $T_{n+1}^{-r}$  and  $\nu_{n+1}$  in the following way:

$$T_{n+1}^r = \begin{pmatrix} \mu_0 & \phi^* \\ \phi & T_n^r \end{pmatrix}; \quad T_{n+1}^{-r} = \begin{pmatrix} \alpha & \gamma^* \\ \gamma & A \end{pmatrix}; \quad \nu_{n+1} = \begin{pmatrix} \mu_{n+1} \\ \nu_n \end{pmatrix}; \quad \phi = \begin{pmatrix} \mu_{-1} \\ \mu_{-2} \\ \vdots \\ \mu_{-n} \end{pmatrix}.$$

After some simple calculations one arrives at

$$\alpha = (\mu_0 - \phi^* T_n^{-r} \phi)^{-1}, \quad \gamma = -T_n^{-r} \phi \alpha,$$

$$A = T_n^{-r} - T_n^{-r} \phi \gamma^*, \quad P_{n+1}^r(0) = -(\alpha \mu_{n+1} + \gamma^* \nu_n).$$

Using the fact that  $T_n^r = LT_n^l L$ , for  $L = \begin{pmatrix} 0 & \cdots & 0 & I \\ 0 & \cdots & I & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdot & \vdots & \vdots \\ I & 0 & 0 & 0 \end{pmatrix}$  one can see that

$$\hat{P}_n^l(x) = x^n (x^{-n} I - [I x^{-1} I \cdots x^{-n+1} I] T_n^{-l} \xi_n) = I - [x^n I x^{n-1} I \cdots I] T_n^{-l} \xi_n$$

$$= I - [I x I \cdots x^n I] L T_n^{-l} L \phi = I - [I x I \cdots x^n I] T_n^{-r} \phi,$$

hence

$$\begin{aligned}
P_{n+1}^r(x) - xP_n^r(x) &= - [\mathbf{I} \ \cdots \ x^n \mathbf{I}] T_{n+1}^{-r} \nu_{n+1} + [x\mathbf{I} \ \cdots \ x^n \mathbf{I}] T_n^{-r} \nu_n \\
&= [x\mathbf{I} \ \cdots \ x^n \mathbf{I}] T_n^{-r} \nu_n - \alpha \mu_{n+1} - \gamma^* \nu_n - [x\mathbf{I} \ \cdots \ x^n \mathbf{I}] (\gamma \mu_{n+1} + A \nu_n) \\
&= [x\mathbf{I} \ \cdots \ x^n \mathbf{I}] \left( T_n^{-r} \phi \gamma^* \nu_n + T_n^{-r} \phi \alpha \mu_{n+1} \right) + P_{n+1}^r(0) \\
&= - [x\mathbf{I} \ \cdots \ x^n \mathbf{I}] T_n^{-r} \phi P_{n+1}^r(0) + P_{n+1}^r(0) = \hat{P}_n^l(x) P_{n+1}^r(0),
\end{aligned}$$

which proves identity (i) of the proposition.

By applying the “ $\hat{\cdot}$ ” operator (introduced at the end of the proposition above) to the identity (i), one obtains (ii). By partitioning the matrix  $T_{n+1}^l$  and applying the same technique as above we obtain (iii) and (iv). Identity (v) is obtained by expressing  $\hat{P}_n^l(x)$  from (iv) and substituting into (i). Identity (vi) is obtained similarly.

In order to prove (vii) let us rewrite identity (v) from the proposition in the following way:

$$\frac{P_{n+1}^r(x)}{x^{n+1}} = \frac{P_n^r(x)}{x^n} (I - P_{n+1}^{l*}(0) P_{n+1}^r(0)) + P_{n+1}^{l*}(x) P_{n+1}^r(0).$$

After multiplying this expression by  $P_n^l(e^{i\theta})W(\theta)$  from the left, substituting  $x = e^{i\theta}$ , integrating and using orthogonality, we arrive at (vii). Identities (viii) and (ix) are proved similarly which concludes the proof of the proposition.  $\blacksquare$

**Note 4.2** Formulas similar to the ones in the proposition above are obtained in a different way and presented in [6, 19, 31].

**Note 4.3** In the classical scalar case  $k = 1$  the expressions above are identical to those obtained in the classical theory of orthogonal polynomials on the unit circle, for example, see [20].

## 5 Kernel Polynomials and the Christoffel–Darboux Formulas

In this section a matrix valued “right” and “left” kernel polynomials are presented for the first time, and the Christoffel–Darboux formula is revisited.

Along with monic orthogonal polynomials, one can introduce orthonormal matrix valued polynomials on the unit circle.

**Definition 5.1** Given families of monic matrix valued orthogonal polynomials defined in (2) and (3), define families  $\{Q_n^r(x)\}_{n=0}^\infty$  and  $\{Q_n^l(x)\}_{n=0}^\infty$  by means of

$$(4) \quad Q_n^r(x) = P_n^r(x) S_n^{-r/2} \quad \text{and} \quad Q_n^l(x) = S_n^{-l/2} P_n^l(x).$$

The orthonormality follows from

$$\langle Q_n^r, Q_n^l \rangle_{r,l} = S_n^{-r,l/2} \langle P_n^r, P_n^l \rangle S_n^{-r,l/2} = S_n^{-r,l/2} S_n^r S_n^{-r,l/2} = I.$$

In order to be able to define an orthonormal family in this fashion, the matrices  $S_n$  have to be positive definite for all  $n$ , which is equivalent to the weight matrix  $W(x)$  being positive definite.

In the lemma below the matrix valued kernel polynomials on the unit circle are presented.

**Lemma 5.2** *Given two families of orthonormal polynomials on the unit circle as defined in (4), denote the “right” and the “left” kernel polynomials of degree  $n$  to be*

$$K_n^r(x, y) = \sum_{i=0}^n Q_i^r(y) Q_i^{r*}(x) \quad \text{and} \quad K_n^l(x, y) = \sum_{i=0}^n Q_i^{l*}(y) Q_i^l(x).$$

Then

$$(i) \quad K_n^r(x, y) = [I \quad \cdots \quad y^n I] T_{n+1}^{-r} \begin{bmatrix} I \\ x^{-1} I \\ \vdots \\ x^{-n} I \end{bmatrix};$$

$$K_n^l(x, y) = [I \quad \cdots \quad y^{-n} I] T_{n+1}^{-l} \begin{bmatrix} I \\ x I \\ \vdots \\ x^n I \end{bmatrix};$$

(ii) *the Christoffel–Darboux formula:*

$$K_n^r(x, y) = \frac{\hat{Q}_{n+1}^l(x) \hat{Q}_{n+1}^{l*}(y) - Q_{n+1}^r(x) Q_{n+1}^{r*}(y)}{1 - x\bar{y}};$$

$$K_n^l(x, y) = \frac{\hat{Q}_{n+1}^{r*}(x) \hat{Q}_{n+1}^r(y) - Q_{n+1}^{l*}(x) Q_{n+1}^l(y)}{1 - x\bar{y}}.$$

**Proof** In order to prove (i) or the right kernel polynomial let us partition  $T_{n+1}^r$  and  $T_{n+1}^{-r}$  in the following fashion:

$$T_{n+1}^r = \begin{pmatrix} T_n^r & \nu_n \\ \nu_n^* & \mu_0 \end{pmatrix} \quad \text{and} \quad T_{n+1}^{-r} = \begin{pmatrix} A & \gamma \\ \gamma^* & \alpha \end{pmatrix},$$

where

$$A = T_n^{-r} + T_n^{-r} \nu_n S_n^{-r} \nu_n^* T_n^{-r}, \quad \gamma = -T_n^{-r} \nu_n S_n^{-r}, \quad \alpha = S_n^{-r}.$$

To ease the notation, denote

$$Y = [I \quad yI \quad \cdots \quad y^{n-1}I] \quad \text{and} \quad X = [I \quad xI \quad \cdots \quad x^{n-1}I].$$

For  $n = 0$  we have  $K_0^r(x, y) = Q_0^r(y) Q_0^{r*}(x) = \mu_0^{-1}$  which agrees with formula (i) in the proposition. To simplify the notation denote the right hand side of expression (i) as  $RHS(n)$ . For the inductive step  $(n - 1) \rightarrow n$  use the partition above as well as

$$[I \quad zI \quad \cdots \quad z^{n-1}I] T_n^{-r} \nu_n S_n^{-r/2} = z^n S_n^{-r/2} - Q_n^r(z)$$

to rewrite  $RHS(n)$  as

$$\begin{aligned}
RHS(n) &= y^n x^{-n} \alpha + y^n \gamma^* X^* + x^{-n} Y \gamma + Y A X^* \\
&= Y(T_n^{-r} + T_n^{-r} \nu_n S_n^{-r} \nu_n^* T_n^{-r}) X^* \\
&\quad - Y T_n^{-r} \nu_n S_n^{-r} x^{-n} - y^n S_n^{-r} \nu_n^* T_n^{-r} X^* \\
&\quad + y^n x^{-n} S_n^{-r} \\
&= Y T_n^{-r} X^* + (y^n S_n^{-r/2} - Q_n^r(y)) (x^{-n} S_n^{-r/2} - Q_n^{*r}(x)) \\
&\quad - x^{-n} (y^n S_n^{-r/2} - Q_n^r(y)) S_n^{-r/2} - y^n S_n^{-r/2} (x^{-n} S_n^{-r/2} - Q_n^{*r}(x)) \\
&\quad + y^n x^{-n} S_n^{-r} = Y T_n^{-r} X^* + Q_n^r(y) Q_n^{*r}(x) \\
&= RHS(n-1) + Q_n^r(y) Q_n^{*r}(x),
\end{aligned}$$

which completes the proof by induction.

In order to derive the Christoffel–Darboux formula (ii) we write the following two recursion relations for orthonormal polynomials:

$$\begin{aligned}
Q_{n+1}^r(t) &= t Q_n^r(t) a + \hat{Q}_n^l(t) b; \quad \hat{Q}_{n+1}^l(t) = \hat{Q}_n^l(t) c + t Q_n^r(t) d; \text{ with} \\
a &= S_n^{r/2} S_{n+1}^{-r/2}, \quad b = S_n^{l/2} P_{n+1}^r(0) S_{n+1}^{-r/2}, \quad c = S_n^{l/2} S_{n+1}^{-l/2} \text{ and } d = S_n^{r/2} P_{n+1}^{*l}(0) S_{n+1}^{-l/2}.
\end{aligned}$$

In matrix form this could be written as

$$\Phi_{n+1}(t) = C(t) \Phi_n(t), \text{ where } \Phi_{n+1}(t) = [Q_{n+1}^r(t); \hat{Q}_{n+1}^l(t)] \text{ and } C(t) = \begin{pmatrix} ta & td \\ b & c \end{pmatrix}.$$

Define  $J = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ , and by the identity (ix) in the proposition in the previous section note that

$$\begin{aligned}
dc^* - ab^* &= S_n^{r/2} P_{n+1}^{*l}(0) S_{n+1}^{-l/2} S_n^{l/2} - S_n^{r/2} S_{n+1}^{-r/2} S_{n+1}^{-r/2} P_{n+1}^{*r}(0) S_n^{l/2} \\
&= S_n^{r/2} (P_{n+1}^{*l}(0) S_{n+1}^{-l} - S_{n+1}^{-r} P_{n+1}^{*r}(0)) S_n^{l/2} = 0; \\
dd^* - aa^* &= S_n^{r/2} P_{n+1}^{*l}(0) S_{n+1}^{-l/2} S_{n+1}^{-l/2} P_{n+1}^l(0) S_n^{r/2} - S_n^{r/2} S_{n+1}^{-r/2} S_{n+1}^{-r/2} S_n^{r/2} \\
&= S_n^{r/2} (P_{n+1}^{*l}(0) S_{n+1}^{-l} P_{n+1}^l(0) - S_{n+1}^{-r}) S_n^{r/2} = -I \\
cc^* - bb^* &= S_n^{l/2} S_{n+1}^{-l/2} S_{n+1}^{-l/2} S_n^{l/2} - S_n^{l/2} P_{n+1}^r(0) S_{n+1}^{-r/2} S_{n+1}^{-r/2} P_{n+1}^{*r}(0) S_n^{l/2} \\
&= S_n^{l/2} (S_{n+1}^{-l} - P_{n+1}^r(0) S_{n+1}^{-r} P_{n+1}^{*r}(0)) S_n^{l/2} = I.
\end{aligned}$$

Hence

$$C(x) J C^*(y) = \begin{pmatrix} x \bar{y} (dd^* - aa^*) & x (dc^* - ab^*) \\ \bar{y} (cd^* - ba^*) & cc^* - bb^* \end{pmatrix} = \begin{pmatrix} -x \bar{y} I & 0 \\ 0 & I \end{pmatrix},$$

which implies that

$$\begin{aligned}\Phi_{n+1}(x)J\Phi_{n+1}^*(y) &= \hat{Q}_{n+1}^l(x)\hat{Q}_{n+1}^{l*}(y) - Q_{n+1}^r(x)Q_{n+1}^{r*}(y) \\ &= \Phi_n(x)C(x)JC^*(y)\Phi_n^*(y) = \hat{Q}_n^l(x)\hat{Q}_n^{l*}(y) - x\bar{y}Q_n^r(x)Q_n^{r*}(y).\end{aligned}$$

Thus,

$$\sum_{k=0}^n Q_k^r(x)Q_k^{r*}(y) = \frac{\hat{Q}_{n+1}^l(x)\hat{Q}_{n+1}^{l*}(y) - Q_{n+1}^r(x)Q_{n+1}^{r*}(y)}{1 - x\bar{y}}.$$

The identity for the left polynomial is proved similarly. ■

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## References

- [1] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*. English translation, Hafner Publishing Co, New York, 1965.
- [2] A. I. Aptekarev and E. M. Nikishin, *The scattering problem for a discrete Sturm-Liouville operator*. Math. Sb. **3**(1983), no. 7, 327–358.
- [3] Yu. M. Berezanskii, *Expansions of eigenfunctions of selfadjoint operators*. Transl. Math. Monographs 17, American Mathematical Society, Providence, RI, 1968.
- [4] C. Brezinski, *The methods of Vorobyev and Lanczos*. Linear Algebra Appl. **234**(1996), 21–41.
- [5] T. S. Chihara, *An introduction to orthogonal polynomials*. Mathematic and its applications 13, Gordon and Breach Science Publishers, New York, 1978.
- [6] P. Delsarte, Y. V. Genin, and Y. G. Kamp, *Orthogonal polynomial matrices on the unit circle*. IEEE Trans. Circuits and Systems **CAS-25**(1978), no. 3, 149–160.
- [7] A. J. Duran, *A generalization of Favard's theorem for polynomials satisfying a recurrence relation*. J. Approx. Theory **74**(1993), no. 1, 83–109.
- [8] ———, *On orthogonal polynomials with respect to positive definite matrix of measures*. Canad. J. Math. **47**(1995), no. 1, 88–112.
- [9] ———, *Markov's theorem for orthogonal matrix polynomials*. Canad. J. Math. **48**(1996), no. 6, 1180–1195.
- [10] ———, *Ratio asymptotics for orthogonal matrix polynomials*. J. Approx. Theory **100**(1999), no. 2, 304–344.
- [11] ———, *Matrix inner product having a matrix symmetric second order differential operator*. Rocky Mountain J. Math. **27**(1997), no. 2, 585–600.
- [12] ———, *Functions with given moments and weight functions for orthogonal polynomials*. Rocky Mountain J. Math. **23**(1993), no. 1, 87–104.
- [13] A. J. Duran and W. Van Assche, *Orthogonal matrix polynomials and higher-order recurrence relations*. Linear Algebra Appl. **219**(1995), 261–280.
- [14] A. J. Duran and P. Lopez, *Orthogonal matrix polynomials: zeros and Blumenthal's theorem*. J. Approx. Theory **84**(1996), no. 1, 96–118.
- [15] A. J. Duran and B. Polo, *Gaussian quadrature formulae for matrix weights*. Linear Algebra Appl. **355**(2002), 119–146.
- [16] R. W. Freund and M. Malhotra, *A block-QMR algorithm for non-Hermitian linear systems with multiple right-hand sides*. Linear Algebra Appl. **254**(1997), 119–157.
- [17] P. A. Fuhrmann, *Orthogonal matrix polynomials and system theory*. Rend. Sem. Mat. Univ. Politec. Torino, 1987, Special Issue, 68–124.
- [18] J. S. Geronimo, *Scattering theory and matrix orthogonal polynomials on the real line*. Circuits Systems Signal Process **1**(1982), no. 3-4, 471–495.
- [19] ———, *Matrix orthogonal polynomials on the unit circle*. J. Math. Phys. **22**(1981), no. 7, 1359–1365.
- [20] Y. L. Geronimus, *Polynomials orthogonal on a circle and their applications*. Amer. Math. Soc. Translations **1954**, no. 104.
- [21] A. El Guennouni, K. Jbilou, and H. Sadok, *The block Lanczos method for linear systems with multiple right-hand sides*. Appl. Numer. Math. **51**(2004), no. 2-3, 243–256.

- [22] G. H. Golub and C. F. Van Loan, *Matrix computations*. Second edition, Johns Hopkins Series in Mathematical Sciences 3, Johns Hopkins University Press, Baltimore, MD, 1989.
- [23] G. H. Golub and R. Underwood, *The block Lanczos methods for computing eigenvalues*. In: *Mathematical software III*, Academic Press, New York, 1977, pp. 364-377.
- [24] M. G. Krein, *Fundamental aspects of the representation theory of Hermitian operators with deficiency index  $(m, m)$* , AMS Translations 2, no. 97, Providence, RI, 1971, 75–143.
- [25] ———, *Infinite  $J$ -matrices and a matrix-moment problem*. Doklady Akad. Nauk SSSR, **69**(1949), 125–128.
- [26] N. N. Lebedev, *Special Functions and their applications*. Prentice-Hall, Englewood Cliffs, NJ, 1965.
- [27] F. Marcellán and I. Rodríguez Gonzàles, *A class of matrix orthogonal polynomials on the unit circle*. Linear Algebra and Appl. **121**(1989), 233–241.
- [28] F. Marcellán and G. Sansigre, *On a class of matrix orthogonal polynomials on the real line*. Linear Algebra and Appl. **181**(1993), 97–109.
- [29] M. P. Mignolet, *Matrix polynomials orthogonal on the unit circle and accuracy of autoregressive models*. J. Comput. Appl. Math. **62**(1995), no. 2, 229–238.
- [30] E. M. Nikishin, *The discrete Sturm-Liouville operator and some problems of function theory*. Trudy Sem. Petrovsk. **10**(1984) 3–77, 237.
- [31] A. Sinap and W. Van Assche, *Orthogonal matrix polynomials and applications*. J. Comput. Appl. Math. **66**(1996), no. 1-2, 27–52.
- [32] G. Szegő, *Orthogonal polynomials*. Fourth edition, American Mathematical Society, Providence, RI, 1975.

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