The magic of the prolate spheroidal functions in various setups

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ABSTRACT

The singular functions for the problem of recovering a time limited function from its Fourier transform in a certain band of frequencies are given, in the simplest case, by the prolate spheroidal wave functions. We explore a number of issues related to this problem, including the effective computation of the corresponding *Slepian* functions for a polar cap on the surface of the Earth. The same method would work for a region on the sphere bounded by two parallels.

Keywords: Time and band limiting, Slepian functions, Numerical Evaluation of prolate spheroidal wave functions

1. INTRODUCTION

The remarkable series of papers,[1–4] by D. Slepian, H. Landau and H. Pollak in connection with the issue of time-and-band limited signals has had a tremendous influence on many areas of engineering, science and mathematics. This work puts on firmer ground some of the pioneering work of C. Shannon. Their starting point were fairly applied aspects of communication theory, optics, lasers, etc. but it was eventually noticed that the ideas were applicable to many other situations.

It is impossible to attempt here a survey of all that work and we just refer the reader to Slepian's paper on the occasion of the J. von Neuman lecture in 1982, see[5] and the references there. One could mention lots of other papers in connection with this work, but we will limit ourselves to mentioning[6,7] and the little known paper.[8]

There are many important aspects in this work, and we cannot possibly do justice to it here. Rather, we pay special attention to the existence of a second order differential operator $\mathbf{D}$ that commutes with the appropriate integral operator $\mathbf{K}$ implementing the time-and-band limiting procedure. An indication of the depth of *this* aspect of the work is the fact that, quite independently, it has appeared in other areas of mathematics. See for instance[9] and the references in that paper.

Our main motivation in concentrating on this commutativity property is practical as well as conceptual. On the practical side this property makes it possible to replace — since the spectrum of $\mathbf{D}$ is simple — the computation of the eigenfunctions of $\mathbf{K}$ for the computation of the eigenfunctions of $\mathbf{D}$, a definitely simpler task from the numerical point of view. Conceptually, it would be important to understand the reason for the existence of a local operator — with simple spectrum — that would commute with a naturally appearing global one like $\mathbf{K}$.

Back in the early 80's one of us was interested in extending this type of results to a different physical/geometrical setup, namely that of functions defined on the unit sphere in three dimensional space. In[10] one finds that both in the case of a polar cap as well as in the case of two symmetrically placed caps (one at each pole) the analysis in the remarkable papers mentioned above can be carried out too. By this we mean that it is possible to exhibit a second order differential operator $\mathbf{D}$ that commutes with the relevant integral operator $\mathbf{K}$ arising from the operation that corresponds to time-and-band limiting. The same applies to the complement, in the sphere, of these regions. If the region in question has less symmetry, then one can always consider the integral operator, but the search for a commuting local operator has proved elusive. For more on this see section 5 below, where we see that some of this old work might now turn out to be useful.

In this paper we show that an interesting method for computing the eigenfunctions of a certain kind of second order differential operators can be used in this spherical cap setup. This method has been used recently in[11] for a number of different purposes.

The first two sections give a quick review of some older work. The hope is that some of this work may someday prove useful, and it is reviewed here for that purpose. We hope that the reader will get from some of this results, many of them quite preliminary in nature, some sense of the richness of the field.

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2. THE MIRACLE IN OTHER SETUPS

The work reported in[9,12–21] is part of a broad search aimed at discovering the reasons underlying the existence of a differential operator that would commute with a "naturally appearing" integral operator. It is probably fair to state that to-date no such conceptual understanding has been achieved.

This search has produced a number of nonstandard situations where (at least an analogue of) the classical situation holds. Some of these examples are discussed below very briefly.

First is the case of,[13] where classical orthogonal polynomials play the role that the exponential functions played in the standard Fourier case. It is important to note that these polynomials are those that are eigenfunctions of a second order difference operator as well as eigenfunctions of a second order differential operator. Polynomials with these two properties were classified in[22] and in[13] the corresponding time-and-band limiting kernels (Christoffel-Darboux) and the corresponding commuting local objects are built. Since spectral and physical space do not coincide in this case we have two versions of all these objects, a continuous one and a discrete one.

On a very different setup one finds in[15] an exploration of the situation when "physical space" is taken to be the set of all permutations of N symbols, i.e. the symmetric group S_N. In this case, except for small values of N, the natural order in the space of representations of this group is not a linear order. The argument is advanced in[15] that a way of obtaining a local operator commuting with the corresponding (global) time-and-band limiting one is to take into account the "dimension" of the corresponding partially orderd set of representations. In all cases previously considered this partial order is a linear one and as argued in[15] this could be the reason while one gets away with second order objects. Of all the results reviewed here, these are the ones that might still reveal some surprises. Both physical and frequency space have a rather nontrivial structure that has not been investigated from the point of view of time-and-band limiting except in.[15] This could also turn out to be useful in some applied areas.

In[16] one finds a complete (and disappointing) classification of all Toeplitz matrices that have a tridiagonal one (with simple spectrum) in its commutator. It would be nice to obtain a similar classification for Hankel matrices. The case of the Hilbert matrix is considered in.[17]

Guided by this search one finds in[18] a proposal for a discrete analog of the Hermite polynomials. It is interesting to note that this work has been pushed a bit further recently by exhibiting a matrix with simple spectrum that commutes with the Discrete Fourier Transform, see.[23] In[18] the property of having a simple spectrum almost works: one of the eigenvalues of the tridiagonal matrix in question is degenerate.

3. THE KDV CONNECTION

In[14] one finds a discussion of some instances of Schroedinger operators of the form

$$\mathbf{L} = \left(\frac{d}{dx}\right)^2 + V(x)$$

such that the appropriate time-and-band integral operator built out of the eigenfunctions of **L** allow for a commuting differential operator. The fact that these examples are very related to celebrated equations like those of Korteweg-deVries (and their master symmetries) gives another indication that this is a very rich subject with connections to many parts of mathematics.

The remarkable relation with a number of topics surrounding the Korteweg- deVries hierarchy of equations, and their master symmetries, appears also in connection with the so called *bispectral problem* considered first in[24] and then reviewed in great detail in.[19] For a number of connections with the theme of time-and-band limiting one can also look at.[12]

The *bispectral property* refers to the property that a family of functions of two variables, one spatial and one spectral, may have: they could be eigenfunctions of two local operators, one on each variable. The remaining variable plays the role of the eigenvalue parameter. In[24] one finds a detailed proof that if both variables run over the reals, and one of the operators is of the type above then all such bispectral cases are related to the KdV story.

There is also a mixed, discrete and continuous, version of this game as well as purely discrete ones. For some of this results one can consult[25,26] and.[27]

In both cases the so called Darboux process plays a very crucial role, allowing one to go from simple bispectral situations to more complicated ones. This process produces for instance all the known examples of *Krall polynomials* and if one is willing to allow more flexibility one gets new kinds of interesting polynomials that should probably be unified by this bispectral approach, see.[28]

4. NUMERICAL CONSIDERATIONS

The accurate and economical numerical computation of the eigenfunctions of **K** is one of the main benefits derived from the existence of a commuting differential operator such as **D**.

Back in 1880, Niven seems to have been the first one to study the prolate spheroidal wave functions by means of an expansion in terms of Legendre functions. This idea has been used for the numerical evaluation of these functions for instance in.[29] This author, and maybe others before him, start from a three term recursion relation satisfied by the coefficients of Niven's expansion. These coefficients are then computed by some form of a continued fraction approach.

Very recently,[11] a proposal has been made that one should use directly the three-term recursion mentioned above to compute the coefficients of Niven's expansion as the eigenvectors for the corresponding tridiagonal semiinfinite matrix. This paper uses this method for several purposes, including interpolation, quadrature rules and differentiation formulae. It is important to realize, that in implementing the ideas in[11] one makes important use of many of the fundamental issues involved in the *bispectral property* of the Legendre polynomials as well as the fact that the prolate spheroidal differential operator is an appropriate perturbation of the Legendre one. Many good things conspire in the right way to give a nice and simple way of computing the eigenfunctions of an integral operator. A similar set of *magical accidents* will be useful in the next section too. For a good source on many aspects of spheroidal wave functions the reader should look into.[30]

5. SLEPIAN FUNCTIONS ON THE SPHERE

One of us is very thankful to Prof. B. Schaffrin for bringing to his attention,[31,32] where the "prolate spheroidal functions" for a spherical cap on the earth (or its complement, or the union of two such complements) are considered in the geodesy community. As mentioned in the introduction this same problem was considered in,[10] as an example in a rather systematic study of situations where the Slepian-Landau-Pollak magic could be extended. One finds in this reference the corresponding commuting differential operator **D** given quite explicitely for functions whose latitude dependence is given by $m = 0$. For other values of m the operator follows directly from the expressions in[10] and it is given in section 6 below.

It is important to note, as was noticed above, that other regions of the earth, like the surface of the oceans, do not lend themselves to this kind of treatment. In this case, one can certainly consider **K** as in,[33] but we have never been able to find a corresponding **D**. On the positive side, a look at[10] will show that the same treatment applies in the case when the gravity field observations are recorded in the complement of two polar caps. We will return to this, and similar cases, in a different publication.

We will see now that we can use the strategy described in[11] to carry out the computation of the eigenfunctions. It is very important to note that we will exploit here, once again, some of the *bispectral properties* enjoyed by a family of polynomials that are just the Legendre ones properly shifted. This, and the fact that the appropriate operator **D** has a special structure vis-a-vis the Legendre polynomials will make this into a good method for the computation of the singular functions of the original problem.

6. NUMERICAL COMPUTATION OF EIGENFUNCTIONS OF THE CORRESPONDING SECOND ORDER DIFFERENTIAL OPERATOR

Consider the following second order differential operator on the interval $[b, 1]$:

$$\mathbf{D} = \frac{d}{dx}\left[(1-x^2)(b-x)\frac{d}{dx}\right] - L(L+2)x - \frac{m^2(b-x)}{1-x^2}.$$

A look at[10] will show that this is the appropriate operator **D** that commutes with the **K** built there when acting on the space of functions whose dependence on ϕ is of the form $e^{im\phi}$. The case of $m = 0$ is written down explicitely in that reference. We will restrict our attention to this case. In order to compute numerically the eigenfunctions F_n (of an appropriate selfadjoint extension) of $\mathbf{D}$ we expand F_n in the basis of the shifted Legendre polynomials, and reduce the problem to the computation of eigenvalues and eigenvectors of certain tridiagonal matrix.

1. Shifted Legendre polynomials

Notation: $b_1 := (1+b)/2$, $b_2 := (1-b)/2$

We define the shifted Legendre polynomials to be solutions of the following second order differential equation:

$$(b-x)(1-x)S_n'' + 2(x-b_1)S_n' - n(n+1)S_n = 0$$

given by:

$$S_n = \frac{1}{b_2^n 2^n n!}\left(\frac{d}{dx}\right)^n [((1-x)(b-x))^n]$$

The following properties of S_n will become useful later:

PROPERTY 1. ***Recurrence relation***

$$xS_n = b_1 S_n + \frac{b_2(n+1)}{2n+1}S_{n+1} + \frac{b_2 n}{2n+1}S_{n-1}, \quad S_{-1} = 0, \quad S_0 = 1$$

PROPERTY 2. ***Differentiation***

$$(1-x)(b-x)S_n' = b_2\frac{n(n+1)}{2n+1}(S_{n+1} - S_{n-1})$$

2. Expansion of F_n along S_n: case $m = 0$

Consider the following eigenfunction/eigenvalue problem:

$$\left(\frac{d}{dx}\left[(1-x^2)(b-x)\frac{d}{dx}\right] - L(L+2)x\right)F_n = \lambda_n F_n \qquad (1)$$

Now, put

$$F_n = \sum_{k=0}^{\infty} a_k S_k$$

and plug into (1). After some manipulations we see that the following tridiagonal symmetric matrix plays an important role:

$$M_{k,k} = k(k+1)(1+b_1) - L(L+2)b_1$$

$$M_{k,k+1} = \frac{b_2(k+1)}{\sqrt{(2k+3)(2k+1)}}[k(k+2) - L(L+2)]$$

$$M_{k+1,k} = \frac{b_2(k+1)}{\sqrt{(2k+3)(2k+1)}}[k(k+2) - L(L+2)]$$

where $k = 0, 1, \ldots$.

Indeed if a denotes the vector with elements a_k, then we obtain an equivalent eigenvector/eigenvalue problem for the tridiagonal matrix M :

$$Ma = \lambda_n a$$

For numerical experiments we choose $b = 1/2$ and $b = 0$, $L = 2$, and $M \in \mathbf{R}^{(n+1)\times(n+1)}$ where $n = 60$. We plot the first 8 eigenfunctions, see Fig.1 and Fig.2. Also, in Fig.3, Fig.4, Fig.5 and Fig.6 we can see how rapidly the coefficients of the expansion decay.

7. ACKNOWLEDGMENTS

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REFERENCES

1. D. Slepian and H. O. Pollak, *Prolate spheroidal wave functions, Fourier Analysis and Uncertainty, I*, Bell System Tech. Journal, Vol. 40, No. 1 (1961), 43–64.
2. H. J. Landau and H. O. Pollak, *Prolate spheroidal wave functions, Fourier Analysis and Uncertainty, II*, Bell System Tech. Journal, Vol. 40, No. 1 (1961), 65–84.
3. H. J. Landau and H. O. Pollak, *Prolate spheroical wave functions, Fourier Analysis and Uncertainty, III*, Bell System Tech. Journal, Vol. 41, No. 4 (1962), 1295–1336.
4. D. Slepian, *Prolate spheroidal wave functions, Fourier Analysis and Uncertainty, IV*, Bell System Tech. Journal, Vol. 43, No. 6 (1964), 3009–3058.
5. D. Slepian, *Some comments on Fourier Analysis, Uncertainty and Modeling*, SIAM Review , Vol. 25, No. 3, July 1983.
6. D. Slepian, *On bandwidth*, Proc. IEEE 63 (1976) pp-292-300.
7. D. Slepian, *Prolate spheroidal wave functions, Fourier Analysis and Uncertainty, V*, Bell System Tech. Journal, Vol. 57, No. 5 (1978), 1371–1430.
8. A.A. Melkman , N-widths and optimal interpolation of time-and-band limited functions, Optimal Estimation in Approximation Theory, C.A. Miccheli and T.J. Rivlin, eds, Plenum, New York, 1977 pp55-68.
9. F. A. Grünbaum, *Band and time limiting, recursion relations and some nonlinear evolution equations* R. Askey et al. (eds) Special functions: Group theoretical aspects and applications, 271-286 1984 D. Reidel Pub. Company.
10. F. A. Grünbaum, L. Longhi, and M. Perlstadt, *Differential operators commuting with finite convolution integral operators: some nonabelian examples*, SIAM J. Appl. Math. **42** (1982), 941–955.
11. H. Xiao,V. Rokhlin and N. Yarvin, *Prolate spheroidal wave functions, quadrature, and interpolation* to appear in Inverse Problems, 2001.
12. F. A. Grünbaum, *Time-band limiting and the bispectral problem*, Comm. Pure Appl. Math. **47** (1994), 307–328.
13. F. A. Grünbaum, *A new property of reproducing kernels of classical orthogonal polynomials*, J. Math. Anal. Applic. **95** (1983), 491–500.
14. F. A. Grünbaum, *Band-time-band limiting integral operators and commuting differential operators*, Algebra i Analiz **8** (1996), 122–126.
15. F. A. Grünbaum, *Some explorations into the mystery of band and time limiting*, Adv. Appl. Math. **13** (1992), 328–349.
16. F. A. Grünbaum, *Eigenvectors of a Toeplitz matrix: discrete version of the prolate spheroidal wave functions*, SIAM J. on Algebraic Discrete Methods **2** (1981), 136–141.
17. F.A. Grünbaum, *A remark on Hilbert's matrix*, Linear Algebra and its applications, 43, 1982 , pp 119-124.
18. F.A. Grünbaum, *Eigenvectors of the discrete Fourier transform: discrete analogue of the Hermite functions* J. of Math. Analysis and Applications, 88, 1982 pp 355-363.
19. F.A. Grünbaum, *Some bispectral musings* in *The Bispectral Problem, editors J. Harnard and A. Kasman* CRM Proceedings and lecture notes, vol 14, 1998 pp 31-45
20. M. Perlstadt, *Chopped orthogonal polynomial expansions – some discrete cases*, SIAM J. Disc. Meth. **4** (1983), 94–100.
21. M. Perlstadt, *A property of orthogonal polynomial families with polynomial duals*, SIAM J. Math. Anal. **15** (1984), 1043–1054.
22. S. Bochner, *Über Sturm-Liouvillesche Polynomsysteme* Math. Z. **29** (1929), 730-736.
23. D. Mugler and S. Clary, *Discrete Hermite functions and the fractional Fourier transform*, Sampta 2001, IEEE, Orlando, Florida pp 303–308.
24. J. Duistermaat and F. A. Grünbaum, *Differential equations in the spectral parameter*, Commun. Math. Phys. **103** (1986), 177–240.
25. F. A. Grünbaum and L. Haine, *A theorem of Bochner, revisited*, in *Algebraic Aspects of Integrable Systems: In memory of Irene Dorfman*, A. S. Fokas and I. M. Gelfand, eds., Progr. Nonlinear Differential Equations **26**, Birkhäuser, Boston (1996), 143–172.
26. F. A. Grünbaum and L. Haine, *Some functions that generalize the Askey–Wilson polynomials*, Comm. in Math. Physics, Vol. 184, No. 1 (1997), 173–202.

27. F. A. Grünbaum and L. Haine, *Bispectral Darboux transformations: An extension of the Krall polynomials*, International Math. Research Notices, No. 8 (1997), 359–392.
28. F. A. Grünbaum, L. Haine and E. Horozov, *On the Krall–Hermite and the Krall–Bessel polynomials*, International Math. Research Notices, No. 19 (1997), 953–966.
29. C. Bouwkamp, *On spheroidal wave functions of order zero*, J. Math. Phys.,MIT, 26 ,1947 , pp 79-92.
30. C. Flammer Spheroidal wave functions Stanford University Press, Stanford, California, 1957.
31. A. Albertella, F. Sanso and N. Sneeuw, *Band-limited functions on a bounded spherical domain: the Slepian problem on the sphere*, Journal of Geodesy (1999) 73, pp 436-447.
32. A. Albertella and N. Sneeuw, *The analysis of gradiometric data with Slepian functions*, Phys. Chem. Earth (A) vol 25, No. 9-11 , pp 667 672 , 2000.
33. Ch. Hwang, *Spectral analysis using orthonormal functions with a case study on the sea surface topography*, Geophys. J. Int 115 , pp 1148- 1160

8. FIGURES

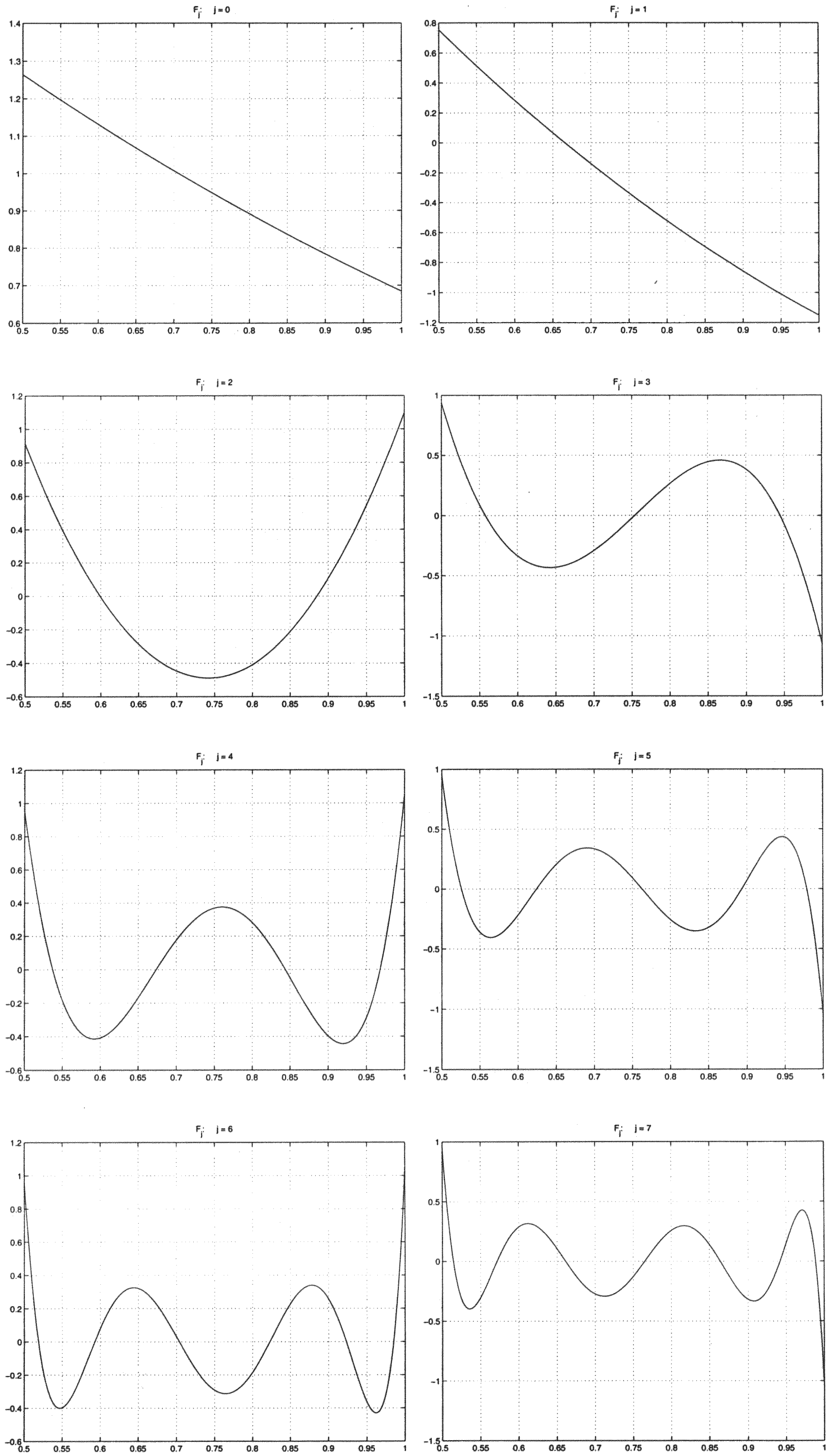

Figure 1. The first 8 eigenfunctions of $\mathbf{D}$ for b = 1/2.

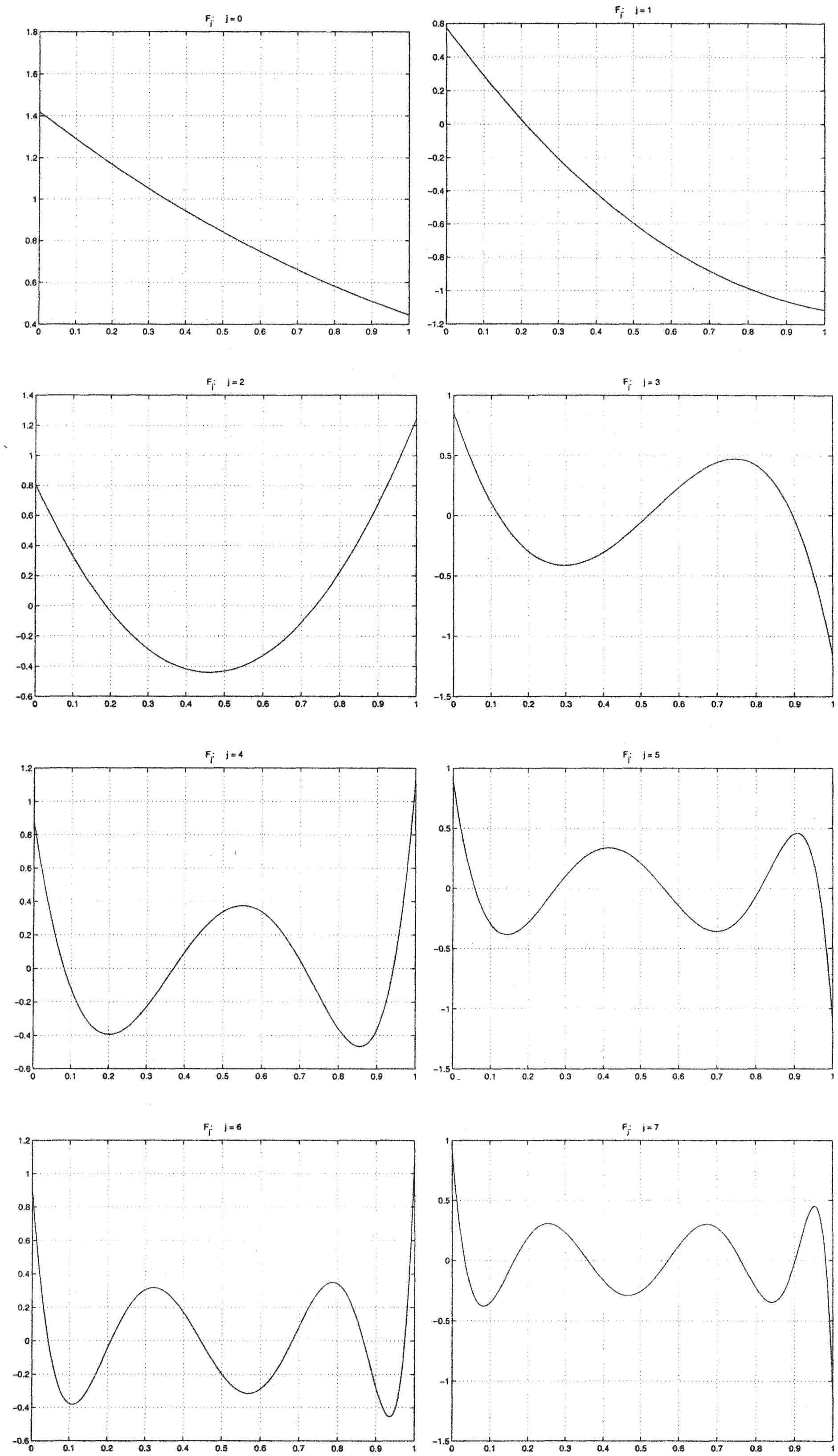

Figure 2. The first 8 eigenfunctions of **D** for b = 0.

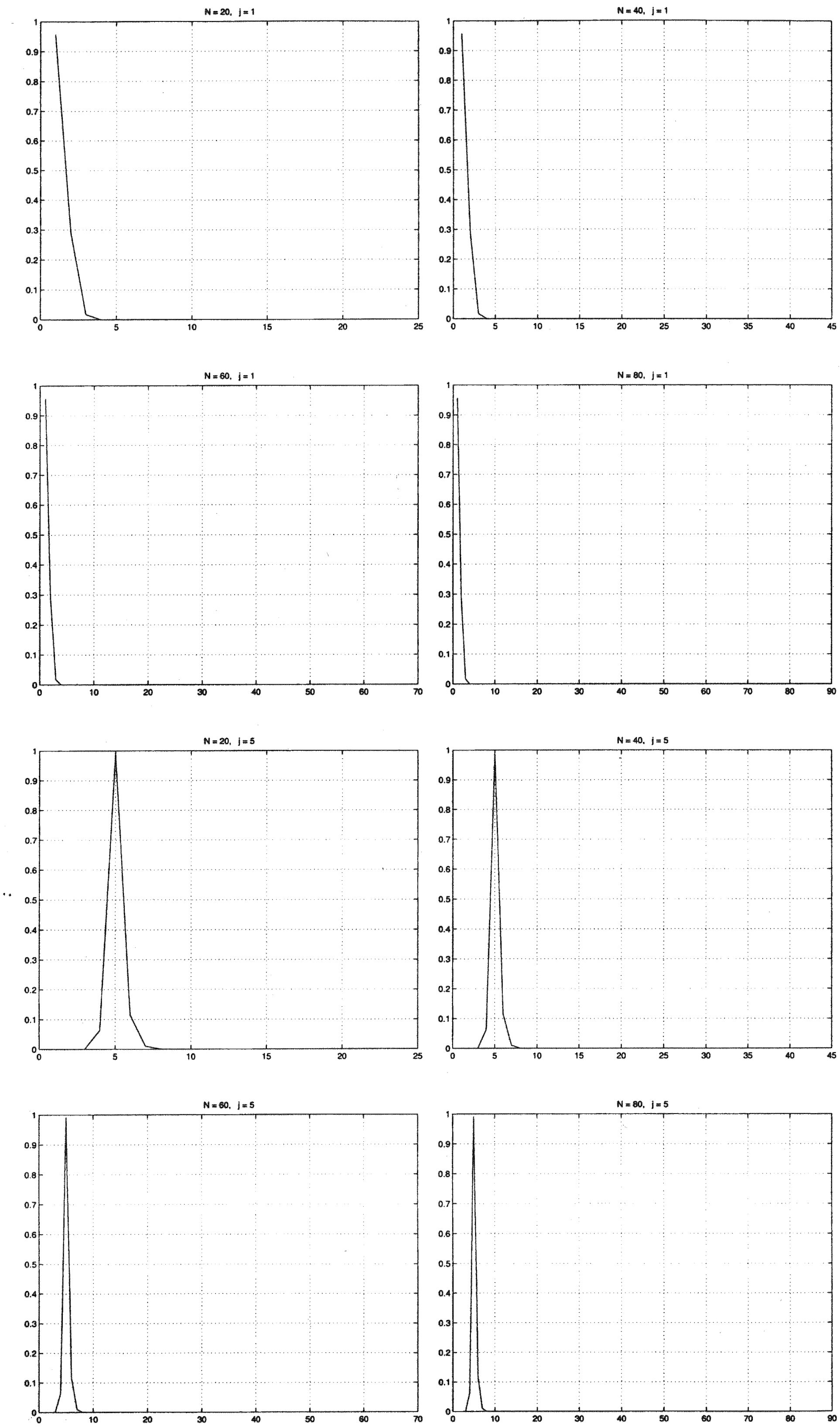

Figure 3. Absolute values of the first N coefficients of the expansion for F_j where j = 1, 5 vs their index; $b = 1/2$.

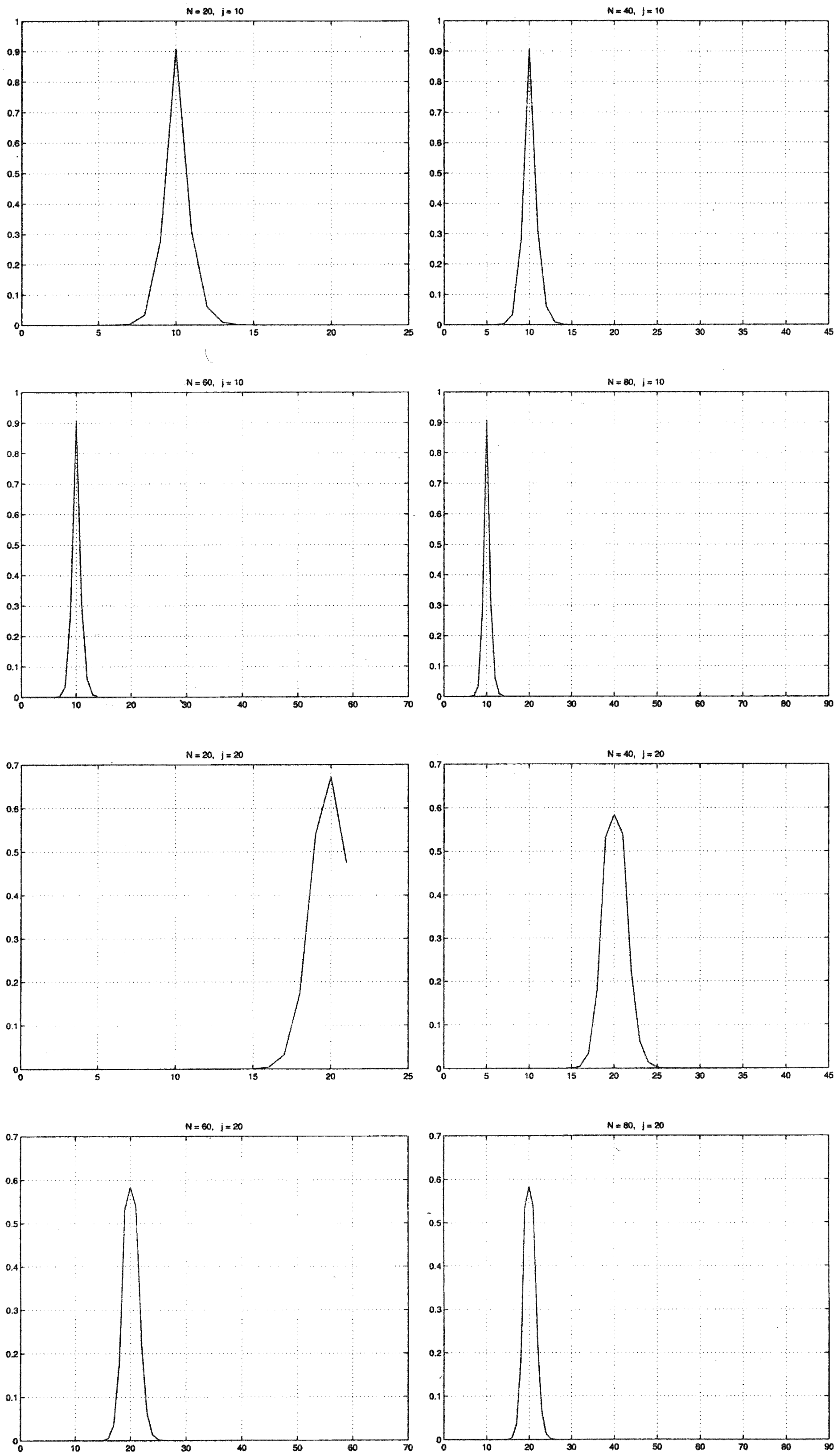

Figure 4. Absolute values of the first N coefficients of the expansion for F_j where j = 10, 20 vs their index; $b = 1/2$.

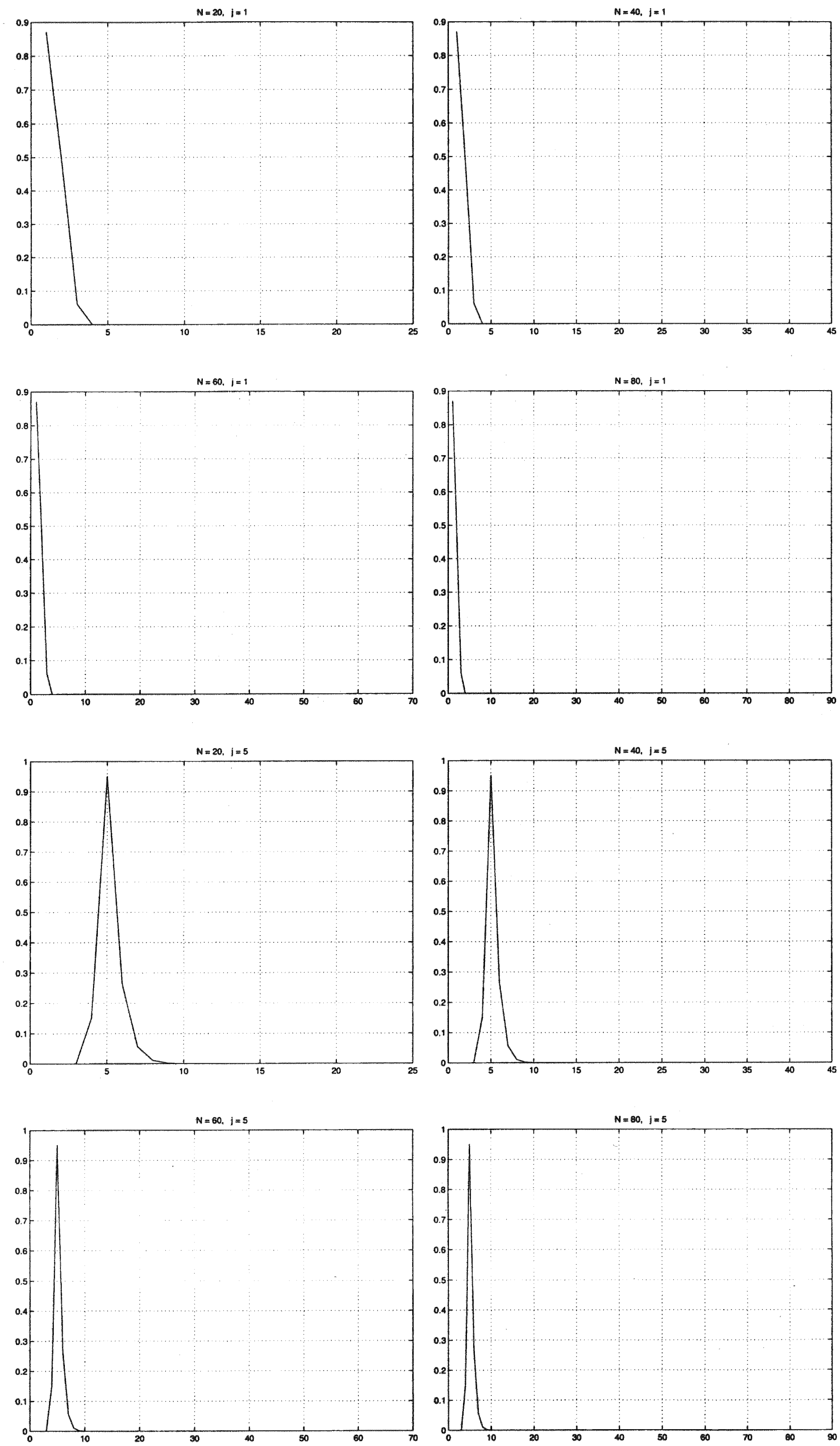

Figure 5. Absolute values of the first N coefficients of the expansion for F_j where j = 1, 5 vs their index; $b = 0$.

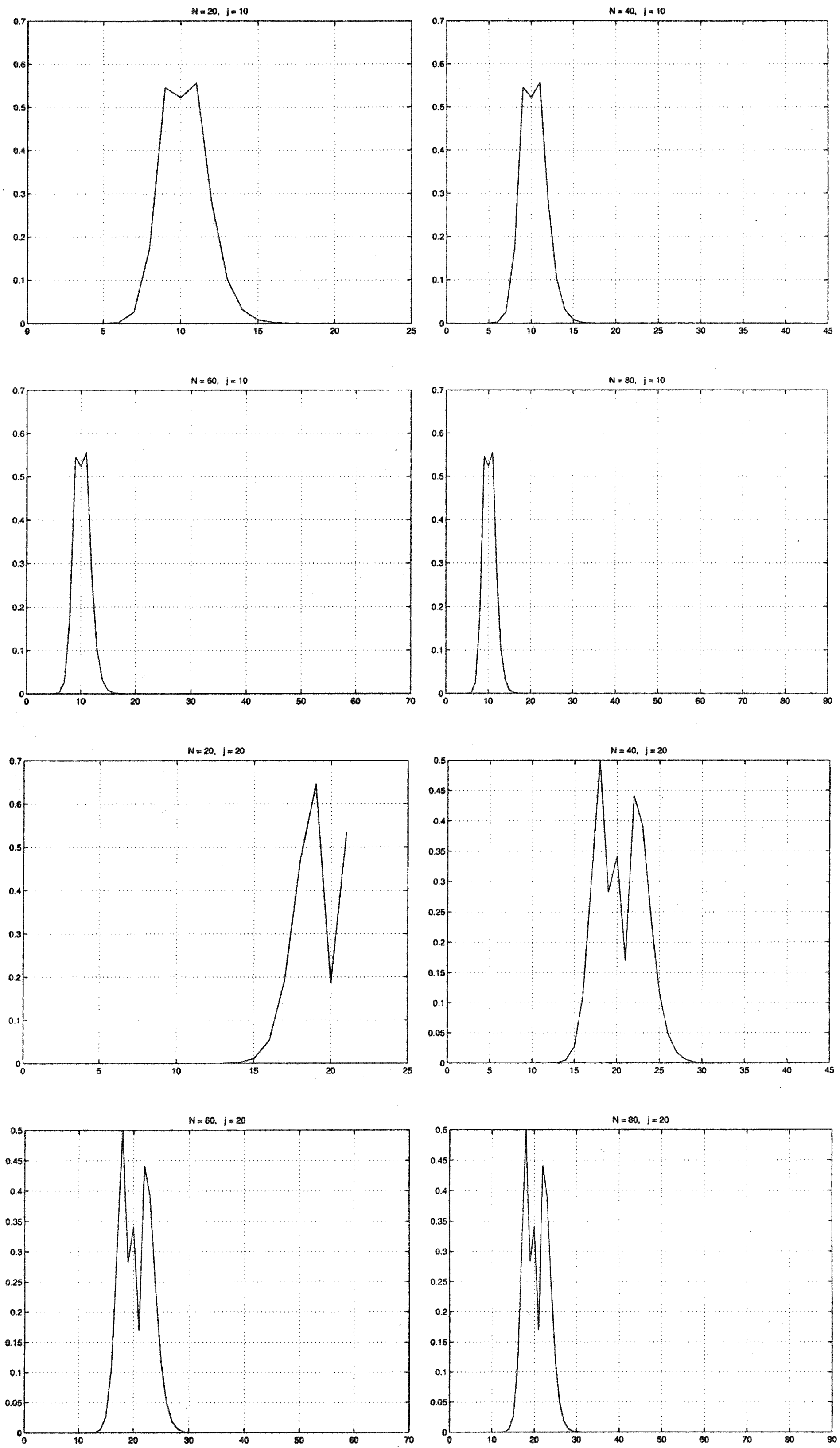

Figure 6. Absolute values of the first N coefficients of the expansion for F_j where j = 10, 20 vs their index; $b = 0$.