

# Construction of Slepian functions on the sphere

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## Classical dual problems

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- Recover a band-limited signal given its values on a certain compact interval in space
- Recover (or approximate) a space-limited signal from the knowledge of its Fourier transform on a certain band of frequencies

## Examples: *physical/frequency* domains – $\mathbf{R}/\mathbf{R}$

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- If a band limited signal

$$f(t) = \int_{-W}^W e^{i\omega t} F(\omega) d\omega$$

is given for  $t \in [-T, T]$ , then it **can** be determined for all  $t \in (-\infty, \infty)$

- If a space limited signal

$$F(\omega) = \int_{-T}^T e^{-i\omega t} f(t) dt$$

is given for  $\omega \in [-W, W]$ , then it **can** be determined for all  $\omega \in (-\infty, \infty)$

## Examples: *physical/frequency* domains – $S_1$ or $S_2/(Z,Z)$

- If a band limited signal

$$f(\theta, \phi) = \sum_{l=0}^L \sum_{m=-l}^l Y_{lm} f_{lm}$$

is given on some compact subset of the sphere  $(\theta, \phi) \in A$ , then it **can** be determined for the entire sphere

- Given Fourier transform of a space limited signal

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm} f_{lm}, \quad (\theta, \phi) \in A$$

on a certain band of frequencies, the signal **cannot** be determined uniquely, but can be approximated

## Formal statement of the problem:

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Determine function  $f$  if

- $f$  is space limited, i.e.  $\mathbf{A}f = f$
- Fourier transform  $\mathbf{F}f$  is known on some compact domain  $B$  in the frequency space, i.e.  $\mathbf{E}f := \mathbf{BFA}f = g$  is given,

where  $\mathbf{A}$  and  $\mathbf{B}$  are operators of restriction to sets  $A$  and  $B$ .

**Goal:** Determine  $f$  given  $\mathbf{E}f = g$

## A natural approach

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Expand the signal  $f$  in a basis with the following properties:

- orthonormal
- complete
- integration rule (generalized Gaussian quadrature)
- can be efficiently constructed

Strategy: use singular vectors of  $E$  as a basis

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- A basis for the space of space-limited functions is given by the eigenvectors of

$$\mathbf{K} := \mathbf{E}^* \mathbf{E} = \mathbf{A} \mathbf{F}^{-1} \mathbf{B} \mathbf{F} \mathbf{A}$$

- Symmetric operator  $\mathbf{K}$  acts on the space  $\mathbf{L}_2(A)$  and its eigenfunctions  $\psi_i$  form a complete, orthonormal basis

Strategy: use right singular vectors of  $E$  as a basis

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To determine  $f$  given  $\mathbf{E}f = g$ , expand

$$f(x) = \sum_{i=0}^{\infty} \alpha_i \psi_i(x) \Rightarrow \mathbf{F}f = \sum_{i=0}^{\infty} \alpha_i \mathbf{F}\psi_i$$

implying

$$\alpha_i = \langle \mathbf{F}f, \mathbf{F}\psi_i \rangle = \langle g, \mathbf{F}\psi_i \rangle_B,$$

since  $\mathbf{BFA}f = g$

## How to compute eigenfunctions of integral operator $\mathbf{K}$ ?

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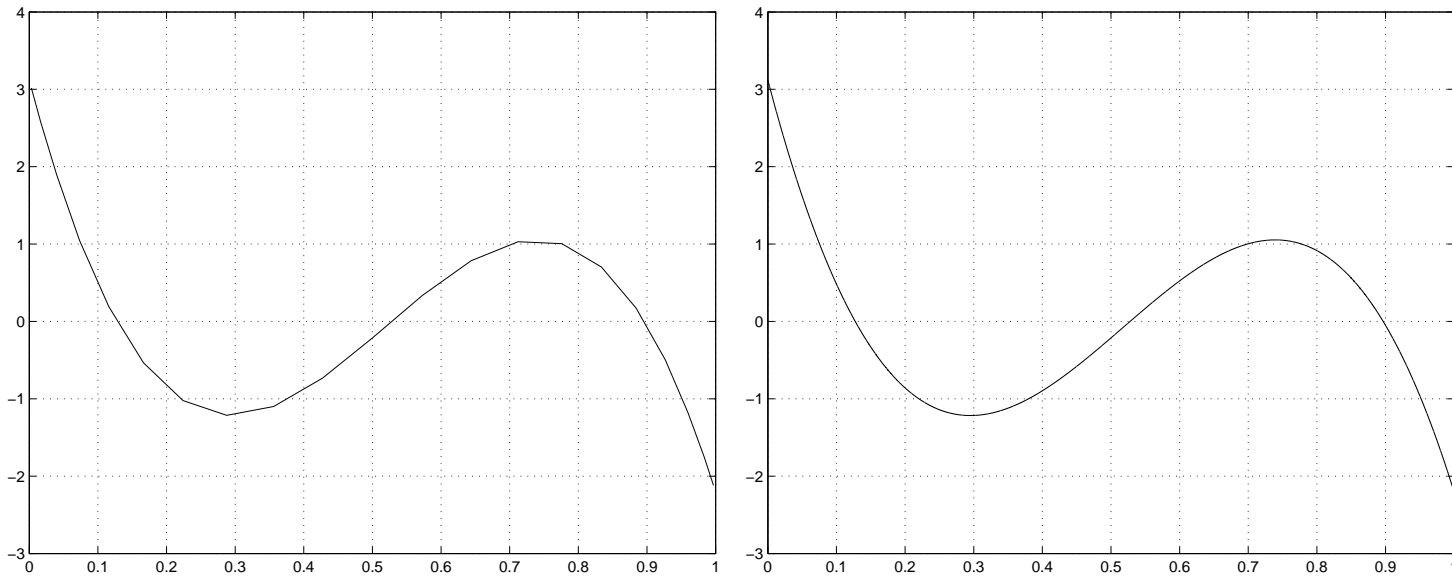
Direct discretization:

- computationally intense – in case of  $S_2$  the size of the problem depends quadratically on the grid size
- eigenfunctions of  $\mathbf{K}$  corresponding to 0 eigenvalue cannot be computed

## Numerical experiments: direct discretization

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$$\psi_3(x) : L = 3, m = 0, b = 0.$$

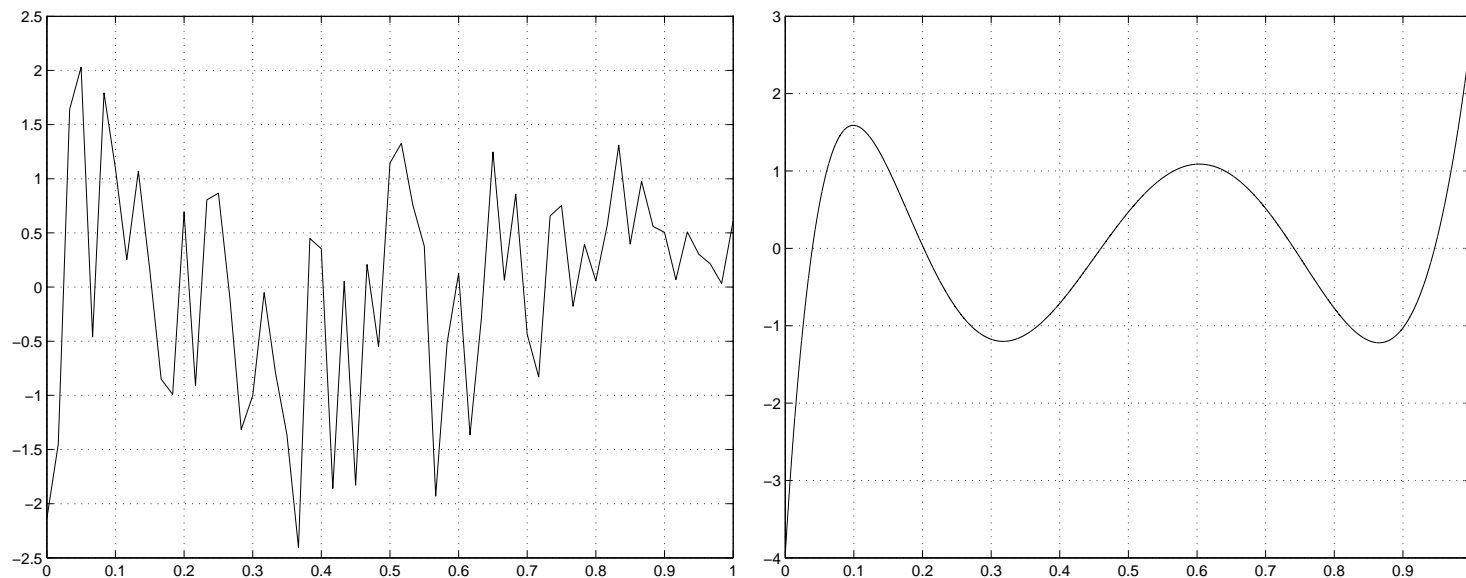


**Left:** obtained by direct discretization of  $\mathbf{K}$  with  $N_\theta = N_\phi = 21$ ;

**Right:** obtained by an alternative method discussed later.

## Numerical experiments: direct discretization

$$\psi_5(x) : L = 3, m = 0, b = 0.$$



**Left:** eigenfunction of  $\mathbf{K}$  corresponding to 0 eigenvalue computed by direct discretization;

**Right:** obtained by an alternative method discussed later

## How to compute eigenfunctions of integral operator $\mathbf{K}$ ?

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In [1], Grünbaum, Longhi and Perlstadt suggested the following strategy:

- Find a differential operator  $\mathbf{D}$  with simple spectrum so that  $\mathbf{DK} = \mathbf{KD}$ . Then eigenfunctions of  $\mathbf{D}$  are also eigenfunctions of  $\mathbf{K}$ .
- Existence of such commuting differential operator is a very interesting and unexplained property of certain integral operators.

## Case of the real line $\mathbf{R}$

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- Integral operator is given by

$$\mathbf{K}(\psi)(x) = \frac{W}{2\pi} \int_{-1}^1 \frac{\sin(W(x-w))}{w-x} \psi(w) dw$$

- A commuting differential operator with simple spectrum

$$\mathbf{D} = (x^2 - 1) \frac{d^2}{dx^2} + 2x \frac{d}{dx} + W^2 x^2,$$

Eigenfunctions of  $\mathbf{D}$  are called Prolate Spheroidal Wave Functions

## Prolate Spheroidal Wave Functions

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Basis of PSWFs  $\{\psi_i(x)\}_{i=0}^{\infty}$  has the following properties:

- Orthonormal, complete, forms a Chebyshev system
- A natural tool for space-limited functions:  $\psi_0$  is the most concentrated signal in  $[-W, W]$ ; among space-limited signals orthonormal to  $\psi_0$ ,  $\psi_1$  is most concentrated, etc.

## How to compute eigenfunctions of $\mathbf{D}$ ?

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In [2], Xiao, Rokhlin and Yarvin suggested the following strategy:

- Observe that Legendre polynomials satisfy differential equation

$$(1 - x^2)L_k''(x) - 2xL_k'(x) + k(k + 1)L_k(x) = 0$$

which is very similar to  $\mathbf{D}$

$$(1 - x^2)\psi_i''(x) - 2x\psi_i'(x) - W^2x^2\psi_i(x) = \lambda_i\psi_i(x)$$

- Expand  $\psi_i(x)$  in the basis of Legendre polynomials

$$\psi_i(x) = \sum_{k=0}^{\infty} \alpha_k^i L_k(x).$$

## How to compute eigenfunctions of $\mathbf{D}$ ?

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- Substitute into  $\mathbf{D}$  to obtain

$$\sum_{k=0}^{\infty} \alpha_k^i [(1 - x^2)L_k''(x) - 2xL_k'(x) - W^2 x^2 L_k(x) - \lambda_i L_k(x)] = 0$$

- Use three term recursion relation

$$(k + 1)L_{k+1}(x) = (2k + 1)xL_k(x) - kL_{k-1}(x)$$

to eliminate  $x^2$  term

## How to compute eigenfunctions of $\mathbf{D}$ ?

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- Derive the symmetric three term recursion relation for  $\alpha_k^i$ 's and write in matrix form

$$M\alpha^i = \lambda_i\alpha^i,$$

where  $\alpha^i = (\alpha_0^i, \alpha_1^i, \dots)$ , and  $M$  is a symmetric tridiagonal matrix.

- Coefficients  $\alpha_k^i$  decay exponentially with respect to  $k$

## Case of the sphere: similar strategy

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- Integral operator is given by

$$\left(\mathbf{K}f\right)(u) = \int_A \left( \sum_{l=0}^L \sum_{m=-l}^l Y_{lm}(u) \overline{Y_{lm}(u')} \right) f(u') du',$$

where

$$u = \left( \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta \right),$$

and  $Y_{lm}(u)$  are the usual spherical harmonics.

## Case of the sphere

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The following was discovered by Grünbaum *et al* in [1]:

- Space of space-limited functions is represented as a sum of subspaces  $H_m$  of functions whose  $\phi$  dependence is given by  $e^{im\phi}$
- Commuting differential operators are constructed on each  $H_m$

$$\mathbf{D} = \frac{d}{dx} \left[ (1 - x^2)(b - x) \frac{d}{dx} \right] - L(L + 2)x - \frac{m^2(b - x)}{1 - x^2},$$

where  $b \leq x = \cos(\theta) \leq 1$ , and  $L$  defines the the band limit.

- Eigenfunctions of  $\mathbf{D}$  are called Slepian functions on the sphere.

## Case of the sphere: computing Slepian functions

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In [3], we adapted the previous strategy in the following way:

- Since spherical harmonics have the form

$$Y_{lm}(x, \phi) = C_{lm}(1 - x^2)^{m/2} \frac{d^m}{dx^m} P_l(x) e^{im\phi},$$

where

$$x = \cos(\theta), \quad C_{lm} = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}$$

and  $P_l(x)$  is regular Legendre polynomial, every  $h \in H_m$  can be written as

$$h(x, \phi) = (1 - x^2)^{m/2} g(x) e^{im\phi}$$

and  $g(x)$  does not have singularities at 1.

## Case of the sphere: computing Slepian functions

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- Note that if  $f(x) = (1 - x^2)^{m/2}g(x)$  satisfies  $\mathbf{D}f = \lambda f$ , where

$$\mathbf{D} = \frac{d}{dx} \left[ (1 - x^2)(b - x) \frac{d}{dx} \right] - L(L + 2)x - \frac{m^2(b - x)}{1 - x^2},$$

then  $g(x)$  satisfies  $\tilde{\mathbf{D}}g = \lambda g$ , where

$$\begin{aligned} \tilde{\mathbf{D}} &= \frac{d}{dx} \left[ (1 - x^2)(b - x) \frac{d}{dx} \right] + 2mx(x - b) \frac{d}{dx} \\ &+ ((m - L)(m + L + 2)x - mb(m + 1)) \end{aligned}$$

## Case of the sphere: computing Slepian functions

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- Shifted Legendre polynomials satisfy differential equation

$$\left[ \frac{d}{dx} \left( (b-x)(1-x) \frac{d}{dx} \right) - k(k+1) \right] L_k(x) = 0$$

which is similar to

$$\begin{aligned} \tilde{\mathbf{D}} &= \frac{d}{dx} \left[ (b-x)(1-x)(1+x) \frac{d}{dx} \right] + 2mx(x-b) \frac{d}{dx} \\ &+ ((m-L)(m+L+2)x - mb(m+1)) \end{aligned}$$

## Case of the sphere: computing Slepian functions

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Denote  $\psi_i(x)$  to be eigefunctions of  $\tilde{\mathbf{D}}$ , then

- Expand  $\psi_i(x)$  in the basis of shifted Legendre polynomials

$$\psi_i(x) = \sum_{k=0}^{\infty} \alpha_k^i L_k(x).$$

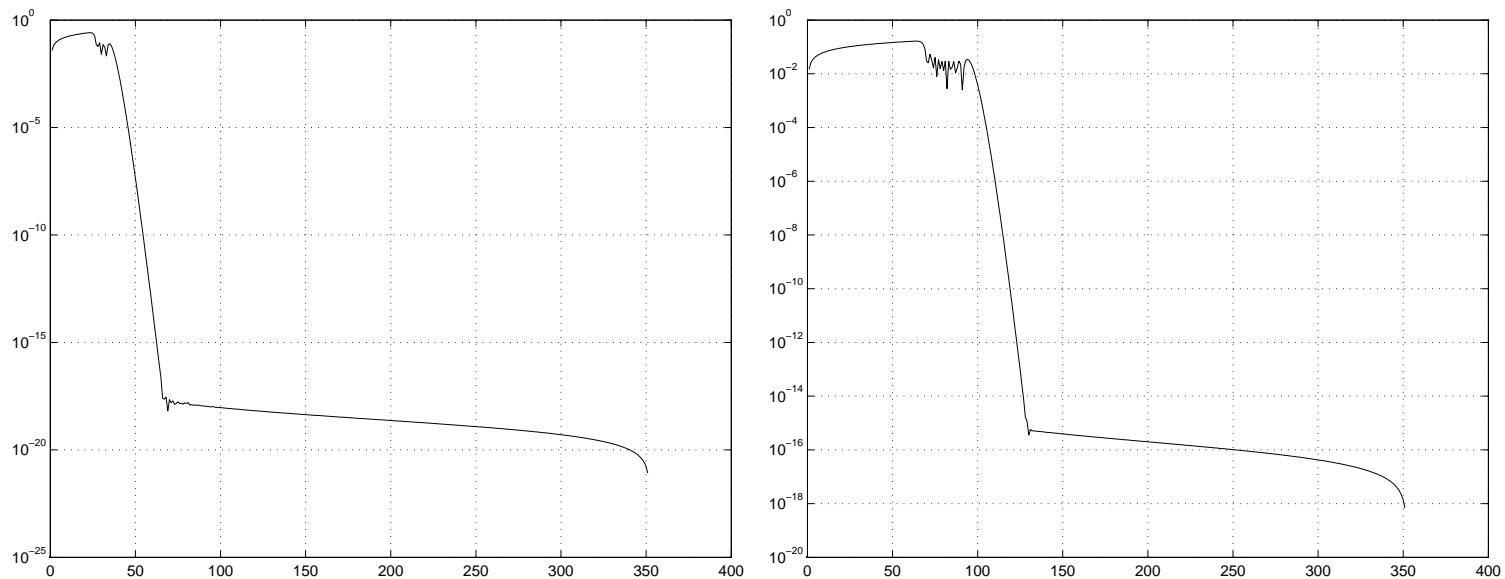
- Derive recursion relation for  $\alpha_k^i$ 's and write in matrix form

$$M\alpha^i = \lambda_i Q\alpha^i,$$

where  $M$  is a symmetric five-diagonal matrix, and  $Q$  is a symmetric tridiagonal matrix

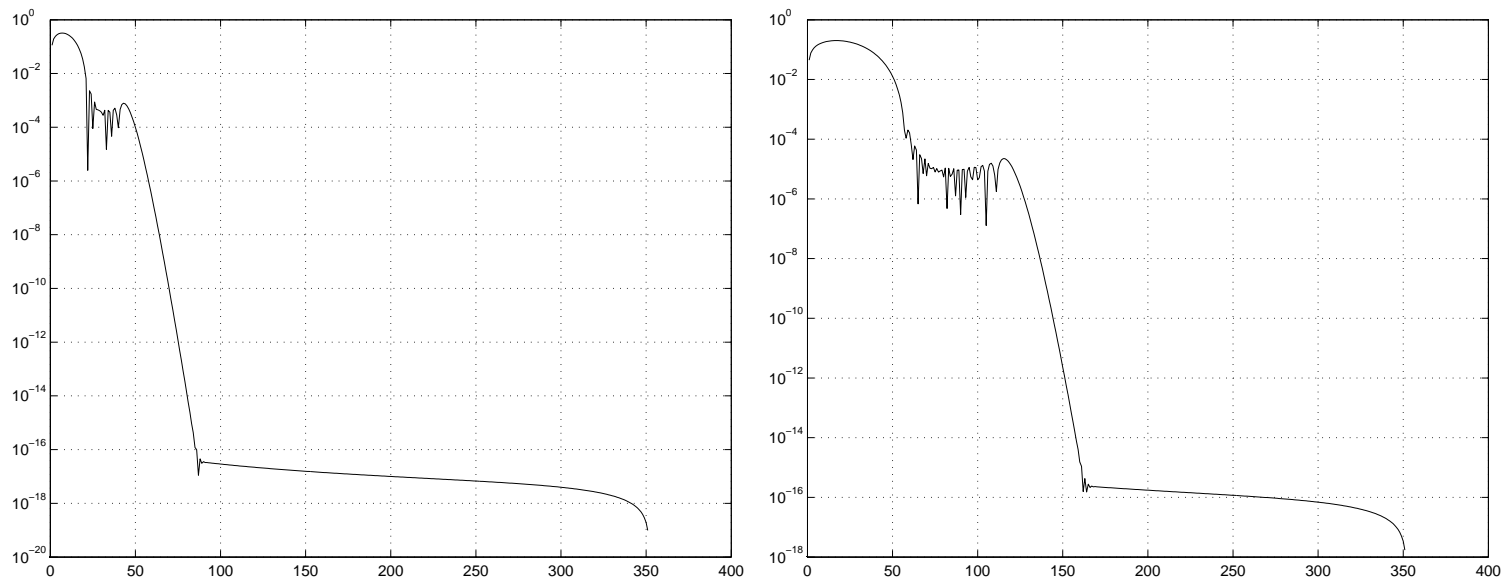
- Observe that coefficients  $\alpha_k^i$  decay very fast with respect to  $k$

## Numerical experiments: decay of expansion coefficients



Absolute values of the first 350 coefficients of the expansions for eigenfunctions  $\psi_{30}$ ,  $\psi_{80}$  vs. their index;  $L = 1$ ,  $m = 1$ ,  $b = 0$ .

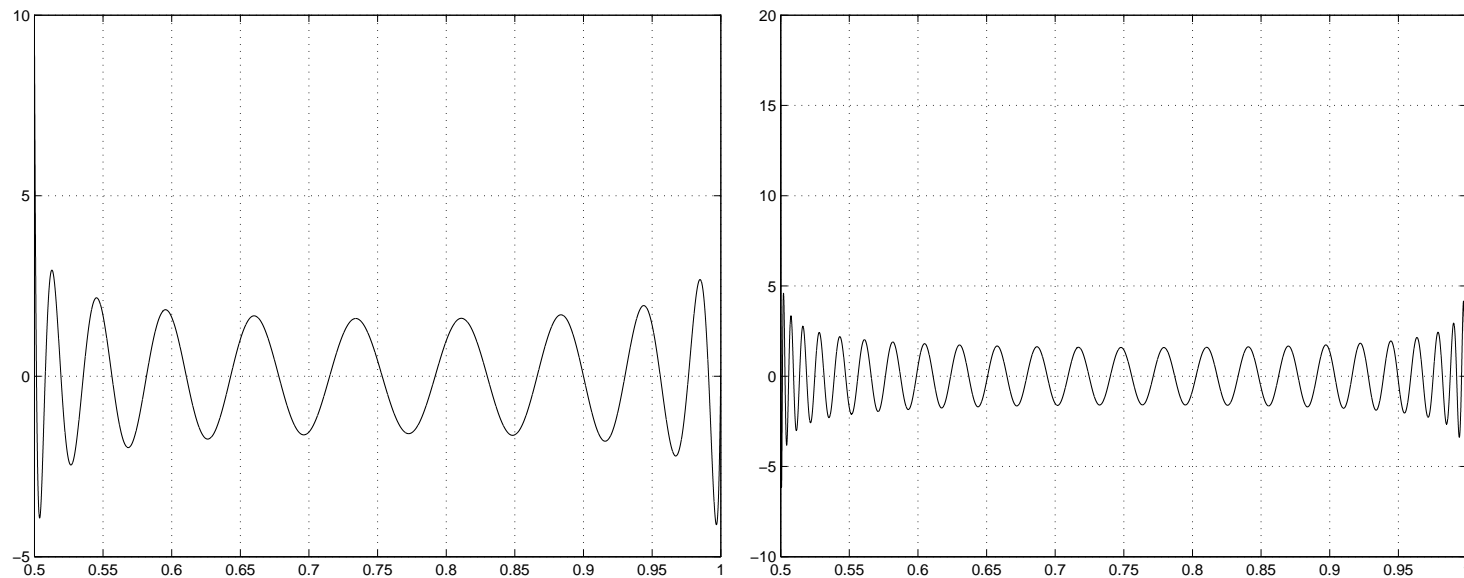
## Numerical experiments: decay of expansion coefficients



Absolute values of the first 350 coefficients of the expansions for eigenfunctions  $\psi_{30}$ ,  $\psi_{80}$  vs. their index;  $L = 7$ ,  $m = 4$ ,  $b = -1/2$ .

# Numerical experiments: Slepian functions

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Eigenfunctions of  $\mathbf{D}$  :  $\psi_{20}$  and  $\psi_{50}$ ,  $L = 5$ ,  $m = 2$ ,  $b = 1/2$ .

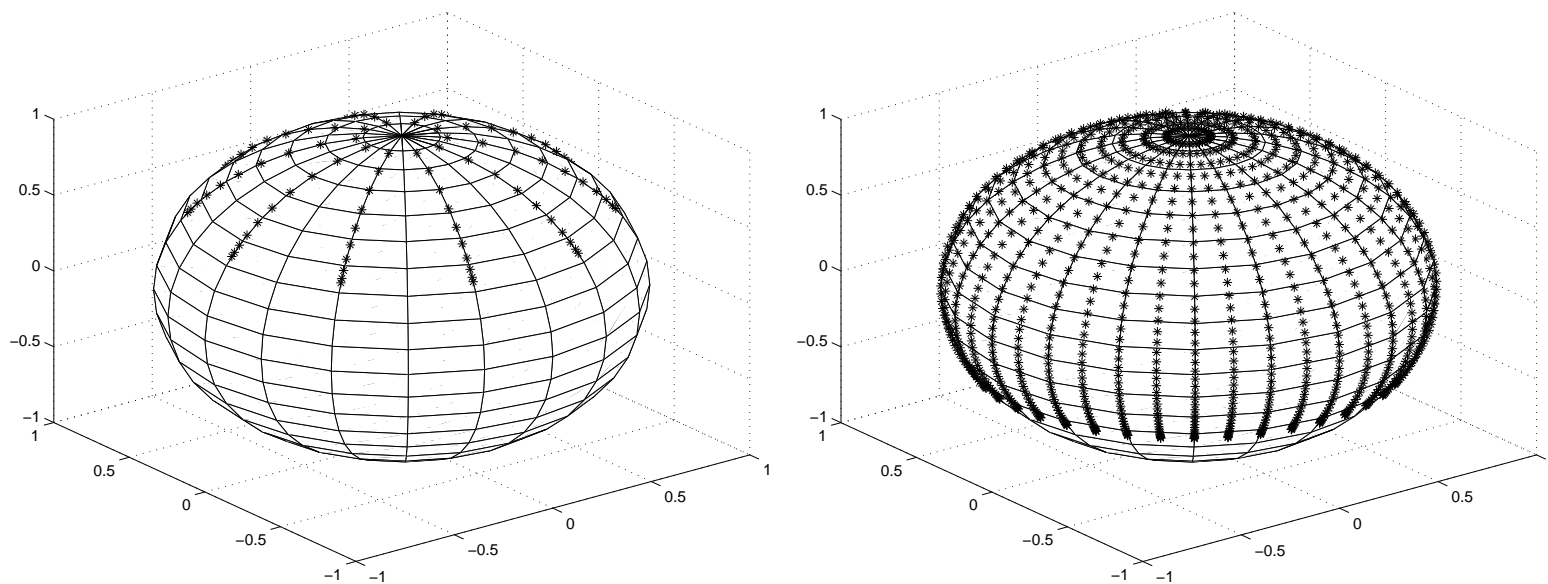
Grid:  $10^4$  points.

## Basis of Slepian functions

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- orthonormal and complete for the space of space-limited functions on the sphere
- forms a Chebyshev system, since Slepian functions are solutions of Sturm-Liouville problem. Generalized Gaussian quadrature rule can be constructed for this basis

## Numerical experiments: generalized Gaussian quadrature



**Left:** nodes for 10-point quadrature,  $b = 1/2$ ,  $L = 5$ ,  $m = 2$ .  
**Right:** nodes for 40-point quadrature,  $b = -1/2$ ,  $L = 1$ ,  $m = 1$ .

## Conclusion

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The basis constructed above has many desirable properties:

- computationally efficient: coefficients of the expansions can be precomputed
- complete and orthogonal on  $L_2(A)$
- forms Chebyshev system, which allows to construct Gaussian quadratures needed for integration on  $A$

## References

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- [1] F. A. Grünbaum, L. Longhi, and M. Perlstadt, *Differential operators commuting with finite convolution integral operators: some non-Abelian examples*, SIAM J. Appl. Math. **42** (1982), 941–955.
- [2] H. Xiao, V. Rokhlin and N. Yarvin, *Prolate spheroidal wave functions, quadrature, and interpolation*, Inverse Problems, 17 (2001), pp. 805-838
- [3] L. Miranian, *Slepian functions on the sphere, generalized Gaussian quadrature rule*, Inverse Problems, 20 (2004), pp. 877-892