Problem Set 9 (due November 12)
MATH 110: Linear Algebra

Each problem is worth 10 points.

PART 1

1. Curtis **p. 161** 10.
2. Curtis **p. 226** 7.
3. Curtis **p. 243** 6.

Solutions: 1. Vandermonde Determinant: Let D be the Vandermonde determinant and let $A = \prod_{i<j}(\zeta_i - \zeta_j)$. Let $x = \zeta_i$. Then by expanding across the ith row of the Vandermonde matrix in the determinant calculation, we see that $D = p(x)$ where p is a polynomial of degree $n-1$. In fact, since ζ_j is a root of p for all $j \neq i$ (since if $x = \zeta_j$ we have two rows of the matrix equal and hence its determinant is 0) we have that $D = c(x - \zeta_1 \cdots (x - \zeta_n)$ with the term $(x - \zeta_i)$ missing in the product. Since i is arbitrary in the argument above, we see that $D = kA$ where k is a polynomial in $\zeta_1, \ldots \zeta_n$. But examining the coefficient of $\zeta_1^{n-1}\zeta_2^{n-2} \cdots 1$ in D, we see that it is either 1 or -1, and therefore k must be either 1 or -1.

2. Similarity of Diagonal matrices: Two similar matrices have the same eigenvalues, and diagonal matrices have as their diagonal entries their eigenvalues. Using these two facts, we see that if the matrices are diagonal and similar, then certainly their diagonals are rearrangements of each other (since they must have the same elements, namely the eigenvalues). Conversely, if the diagonal elements are rearrangements of each other, then the matrices are similar because they represent the same linear transformation, with respect to the same basis except with the basis elements rearranged.

3. We are assuming that V is finite dimensional. Suppose that $\lambda_v = \lambda_w$. Then for any linear functional f, we have that $f(v) = f(w)$. Therefore $f(v - w) = 0$. Let $B = \{v_1, \ldots, v_n\}$ be a basis for V and $C = \{f_i\}$ be the dual basis of B. Then in particular, $f_i(v - w) = 0$ for every i, and so if we write $v - w = \sum_{i=1}^{n} c_i v_i$, we have that $c_i = 0$ for every i. Therefore $v - w = 0$ and $v = w$. Thus we have that our map is $1-1$, and it follows that it is an isomorphism, since $dim(V^*)^*) = dim(V)$. To see this, let $D = \{\mu_i\}$ be the dual basis of C, that is $\mu_i(f_j) = 0$ if $j \neq i$ and 1 if $j = i$. Notice that in fact,

$mu_i = \lambda_{v_i}$. This is because $\lambda_{v_i}(f_j) = f_j(v_i)$ which is 1 if $i = j$ and 0 if $i \neq j$. Then $\lambda_{v_1}, \ldots, \lambda_{v_n}$ is a basis for $(V^*)^*$. Therefore the spaces have the same dimension (since they have bases of the same size).

PART 2

Problem 1(20)

Let V be a real vector space of functions spanned by the set of real values functions $\{e^x, xe^x, x^2e^x, e^{2x}\}$ and let T be the linear transformation $T : V \to V$ defined by $T(f) = f'$, the derivative of f. Find the Jordan canonical form of T.

Solution: The characteristic polynomial is $h(x) = (x-1)^3(x-2)$ which is also the minimal polynomial (this requires a bit of checking). It follows, by computing the companion matrices, that the Jordan canonical form is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Problem 2(10)

Prove that if V is isomorphic to W then V^* is isomorphic to W^*. Is the converse true (prove or give a counterexample)?

Solution: I meant for you to assume that V, W are finite dimensional, in which case the claim is true (and the converse), by the same arguments as number 3 in part 1.

Problem 3 (10)

a) Let $T : R \to R$ be a linear transformation. Show that $T(x) = cx$ where $c \in R$ is some constant.

Proof: We will use the fact that $L(R, R)$ is isomorphic to $M_{1,1}(R)$ (the 1×1 real matrices). Given this fact, the claim is clear since using the standard basis to represent T, we find that $Ax = cx$.

b) Let $T : R_2 \to R$ be a linear transformation. Show that $T(x, y) = c_1x + c_2y$.

Proof: Same as above, except now we find that $L(R_2, R)$ is isomorphic to $M_{1,2}(R)$. Again, using the standard basis, we find that Av^t (now v is a 2×1 vector $v = [xy]$) is just $c_1x + c_2y$.

c) Generalize parts a) and b) to a linear transformation of the form $T : R_n \to R$.

Proof: Same as above. We will find that if $x = (x_1, \ldots, x_n) \in R_n$ then $T(x) = c_1x_1 + \cdots + c_nx_n$.

d) Show that every plane through the origin in R_3 may be identified with the null space of an element in $(R_3)^*$.

Proof: A plane through the origin in R_3 has the form $ax + by + cz = 0$. Consider an element T in $(R_3)^*$. This is a linear functional, which is an element of $L(R_3, R)$, and so $T(x) = c_1x + c_2y + c_3z$. The null space of T consists precisely of those vectors (x, y, z) such that $c_1x + c_2y + c_3z = 0$.

Problem 4 (10)

Let $T : M_{n,n}(R) \to M_{n,n}(R)$ be a linear transformation from the vector space of $n \times n$ matrices over R into itself, where $T(A) = A^t$. Find the minimal polynomial of T.

Solution: The minimal polynomial is $m(x) = x^2 - 1$ unless $T = cI$. This is because $T^2 = I$, and the minimal polynomial has degree 1 only if $T = cI$.

PART 3 - Optional Problem

Prove the lights and switches result combinatorially (i.e., without using linear algebra).

Proof: Let G be a graph on n vertices and let O be those vertices in G with odd degree (i.e. an odd number of edges adjacent to them). Assume by induction that we can flip the lights of any graph with $n - 1$ vertices. The algorithm for flipping all the lights in G is (check that it works!):

1) Flip every vertex.

2) Flip all the lights in the graphs $G - x$ where $x \in O$ (done once for every x in O).