Problem Set 11 Solutions
MATH 110: Linear Algebra

Each problem is worth 10 points.

PART 1

1. Curtis **p. 276** 2.
2. Curtis **p. 276** 4.
3. Curtis **p. 276** 9.
4. Curtis **p. 288** 4.

PART 2

Problem 1(20)

Show that for any $1 \times n$ row matrix X, and any $n \times n$ matrix A, we have that

$$XAX^t = XBX^t$$

where B is the symmetric matrix $B = \frac{1}{2}(A + A^t)$.

Proof: XAX^t is a 1×1 matrix, so $XAX^t = (XAX^t)^t = XA^tX^t$. Therefore, $XAX^t = \frac{1}{2}(XAX^t + XA^tX^t) = XBX^t$.

Problem 2(10)

Let $T : V \to V$ be a symmetric transformation on a real Euclidean space V with quadratic form $Q(x) = (T(x), x)$. Prove that if Q does not change sign sign on V (i.e. $Q(x) \geq 0\ \forall x \in V$ or $Q(x) \leq 0\ \forall x \in V$) then $Q(x) = 0$ implies that $T(x) = 0$. (Hint: consider the function $p(t) = Q(x + ty)$ and show that $p'(0) = 0$).

Proof: Consider $p(t)$ as suggested in the hint. In fact, $p(t) = Q(x+ty) = (T(x + ty), x + ty) = (T(x), x) + t(T(x), y) + t(T(y), x) + t^2(T(y), y)$ from the properties of the inner product and the linearity of T. We can thus write $p(t) = at + bt^2$ for some constants a, b depending on x, y. Therefore $p'(t) = (T(x), y) + (T(y), x) + 2t(T(y), y)$ and $p'(0) = (T(x), y) + (x, T(y))$. T is symmetric so the two terms in the sum are equal. Now suppose $p(t) \geq 0$ for every t and since $p(0) = 0$, it follows that $p'(0) = 0$ (the same works for $p(t) \leq 0$). It follows that $(T(x), y) = 0$ for every y, and so $T(x) = 0$.

Problem 3 (10)

a) Prove that the set of symmetric matrices forms a subspace of dimension $\frac{n(n+1)}{2}$ in the vector space of real $n \times n$ matrices (which has dimension n^2).

Proof: Consider the standard basis for the vector space of real $n \times n$ matrices. We can use it to build a basis for the symmetric matrices, by consider $A + A^t$ for a basis element A. This gives a basis of size $\frac{n(n+1)}{2}$.

b) Prove that the set of skew-symmetric matrices forms a subspace of dimension $\frac{n(n-1)}{2}$ in the vector space of real $n \times n$ matrices (which has dimension n^2).

Proof: Same as part (a) except that you can use the basis elements which contain only an element on the diagonal, thus removing n basis elements.

Problem 4 (10)

Let T be a symmetric linear transformation on a real Euclidean space. Prove that T has a "square root", i.e. there is a symmetric linear transformation S such that

$$T = S^2.$$

Proof: The fact that T is symmetric means that $T = C^{-1}DC$ where D is a diagonal matrix. Let $D^{\frac{1}{2}}$, the square root of D, be the diagonal matrix consisting of the square roots of the entries of D. Notice that $C^{-1}DC = C^{-1}D^{\frac{1}{2}}CC^{-1}D^{\frac{1}{2}}C$ because $CC^{-1} = I$. Therefore $T = S^2$ where $S = C^{-1}D^{\frac{1}{2}}C$.

Problem 5 (10)

Define the index of a real symmetric matrix A to be the number of strictly positive eigenvalues of A minus the number of strictly negative eigenvalues of A. Let $Q_A(x)$ and $Q_B(x)$ be the quadratic forms associated with A and B, and suppose that

$$Q_A(x) \leq Q_B(x)$$

for all vectors x. Prove that the index of A is less than or equal to the index of B.

Proof: We will use the result from number 8, page 276 in Curtis (known as Sylvester's theorem). Let A_+ be the subspace consisting of those vectors for which $Q_A(x) > 0$ and let B_+ be the subspace consisting of those vectors for which $Q_B(X) > 0$. Notice that if $v \in A_+$, then $v \in B_+$, and so the dimension of B_+ is at least as large as that of A_+. The reverse is true for A_- and B_-, those subspaces corresponding to vectors for which the quadratic forms are negative. The result follows.

Optional Problem

Let $T : V \to V$ be a symmetric linear transformation on a real Euclidean space V and let $Q(x) = (T(x), x)$. Assume that there is an extremum (maximum or minimum) at u for $Q(x)$ among all the values that Q takes on the unit sphere (i.e. $Q(u)$ is either a max or a min amongst all the values $Q(x)$ with $(x, x) = 1$). Show that u is an eigenvector for T.